M I a

KUMON

Points and Lines 1

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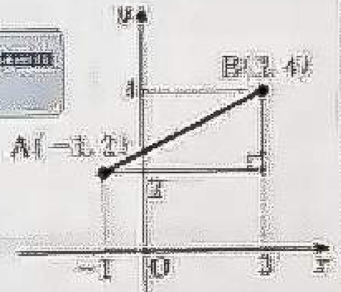
1. Find the distance between the two given points.

Ex. A(−1, 2), B(3, 4)

[Sol] $AB=\sqrt{(3+1)^2+(4-2)^2}$ ← The Pythagorean Theorem

$=\sqrt{20}$

$=2\sqrt{5}$

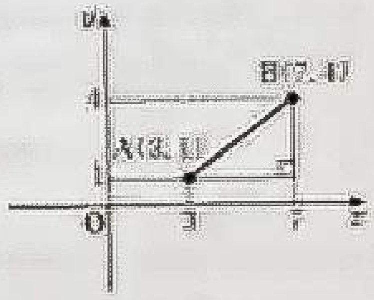


(1) A(3, 1), B(7, 4)

[Sol] $AB=\sqrt{(7-3)^2+(4-1)^2}$

$=\sqrt{25}$

$=5$

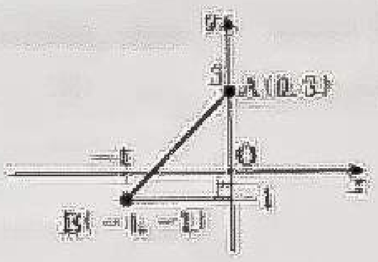


(2) A(0, 3), B(−4, −1)

[Sol] $AB=\sqrt{(-4-0)^2+(-1-3)^2}$

$=\sqrt{32}$

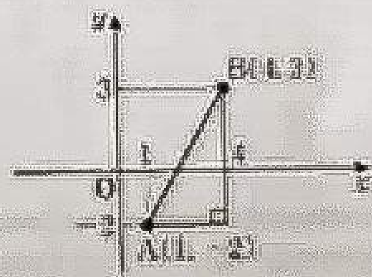
$=4\sqrt{2}$



(3) A(1, −2), B(4, 3)

[Sol] $AB=\sqrt{(4-1)^2+(3+2)^2}$

$=\sqrt{34}$



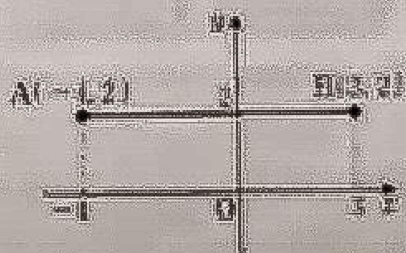
(4) A(−4, 2), B(3, 2)

[Sol] $AB=\sqrt{(3+4)^2+(2-2)^2}$

$=\sqrt{49}$

$=7$

[Alternative Solution
$AB=3+4$
$=7$]



M I b

2. Find x when the distance between two points $A(2, -3)$ and $B(x, 5)$ is 10.

[Sol] Since $AB=10$,

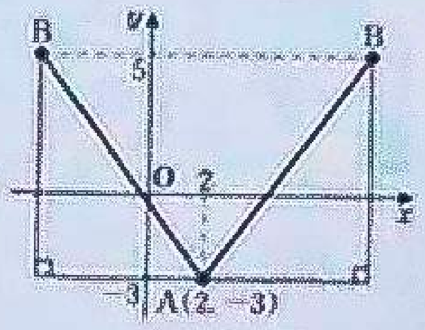
$$\sqrt{\boxed{(x-2)^2+(5+3)^2}}=10$$

Squaring both sides and simplifying

$$\therefore\ x^2-4x-32=0$$

$$(x+4)(x-8)=0$$

$$\therefore\ x=-4,\ 8$$



3. Find x when the distance between two points $A(x, 2)$ and $B(5, 5)$ is 5.

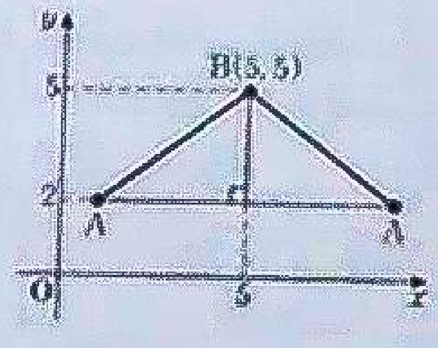
[Sol] Since $AB=5$,

$$\sqrt{(5-x)^2+(5-2)^2}=5$$

$$\therefore\ x^2-10x+9=0$$

$$(x-1)(x-9)=0$$

$$\therefore\ x=1,\ 9$$



. Find y when the distance between two points $A(-1, 3)$ and $B(2, y)$ is $3\sqrt{5}$.

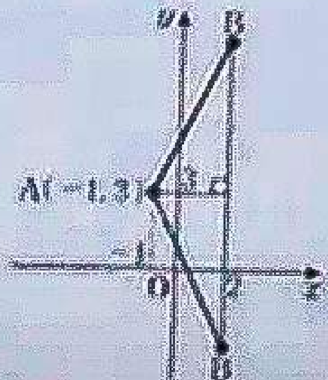
Sol] Since $AB=3\sqrt{5}$,

$$\sqrt{(2+1)^2+(y-3)^2}=3\sqrt{5}$$

$$\therefore\ y^2-6y-27=0$$

$$(y+3)(y-9)=0$$

$$\therefore\ y=-3,\ 9$$



enerally, the following formula is true.

Distance Formula

he distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

lso, the distance between origin O and point $A(x_1, y_1)$ is

$$OA=\sqrt{x_1^2+y_1^2}$$

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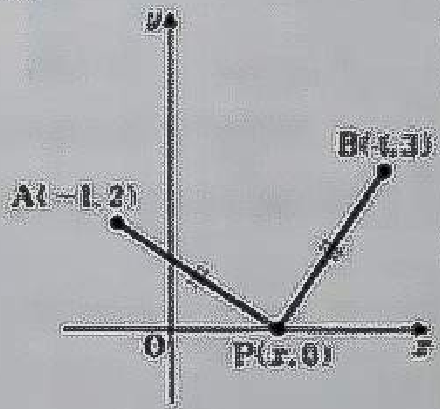
Ex. Find the coordinates of point P on the x-axis such that it is equidistant from two points $A(-1, 2)$ and $B(4, 3)$.

[Sol] Let the coordinates of point P be $(x, 0)$.

$$AP = \sqrt{(x+1)^2+(0-2)^2}$$
$$= \sqrt{x^2+2x+5}$$
$$BP = \sqrt{(x-4)^2+(0-3)^2}$$
$$= \sqrt{x^2-8x+25}$$

Since $AP = BP$, i.e. $AP^2 = BP^2$,

$$x^2+2x+5 = x^2-8x+25$$
$$\therefore x = 2$$
$$\therefore P(2, 0)$$



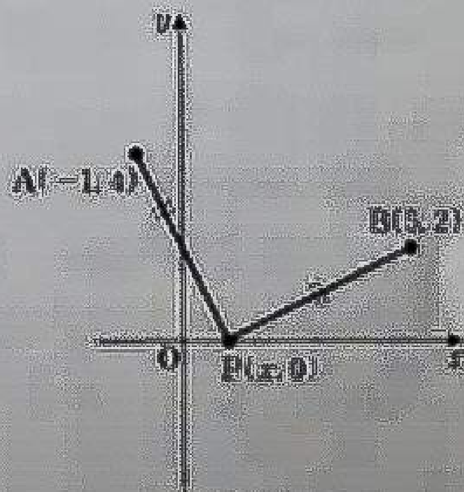
1. Find the coordinates of point P on the x-axis such that it is equidistant from two points $A(-1, 4)$ and $B(5, 2)$.

[Sol] Let the coordinates of point P be $(x, 0)$.

$$AP = \sqrt{(x+1)^2+(0-4)^2}$$
$$= \sqrt{x^2+2x+17}$$
$$BP = \sqrt{(x-5)^2+(0-2)^2}$$
$$= \sqrt{x^2-10x+29}$$

Since $AP = BP$, i.e. $AP^2 = BP^2$,

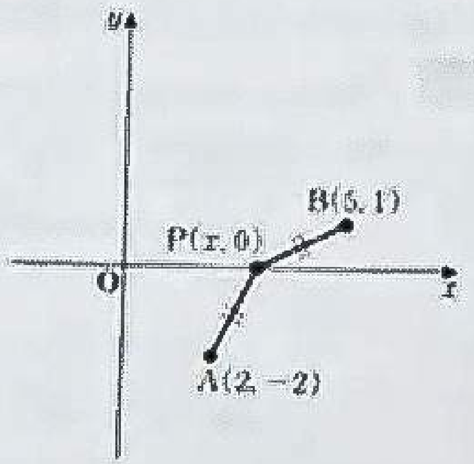
$$x^2+2x+17 = x^2-10x+29$$
$$\therefore x = 1$$
$$\therefore P(1, 0)$$



M2b

2. Find the coordinates of point P on the x-axis such that it is equidistant from two points $A(2, -2)$ and $B(5, 1)$.

[Sol] Let the coordinates of point P be $(x, 0)$.



$$AP=\sqrt{(x-2)^2+(0+2)^2}$$
$$=\sqrt{x^2-4x+8}$$
$$BP=\sqrt{(x-5)^2+(0-1)^2}$$
$$=\sqrt{x^2-10x+26}$$

Since $AP=BP$, i.e. $AP^2=BP^2$,

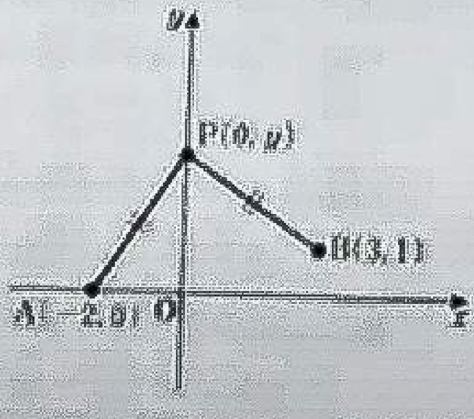
$$x^2-4x+8=x^2-10x+26$$

$\therefore\ x=3$

$\therefore\ P(3, 0)$

Find the coordinates of point P on the y-axis such that it is equidistant from two points $A(-2, 0)$ and $B(3, 1)$.

ol] Let the coordinates of point P be $(0, y)$.



$$AP=\sqrt{(0+2)^2+(y-0)^2}$$
$$=\sqrt{y^2+4}$$
$$BP=\sqrt{(0-3)^2+(y-1)^2}$$
$$=\sqrt{y^2-2y+10}$$

Since $AP=BP$, i.e. $AP^2=BP^2$,

$$y^2+4=y^2-2y+10$$

$\therefore\ y=3$

$\therefore\ P(0, 3)$

Points and Lines 1

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Ex. Given $\triangle ABC$ with vertices $A(-1, 0)$, $B(2, 4)$ and $C(3, 2)$, show that $\triangle ABC$ is a right-angled triangle.

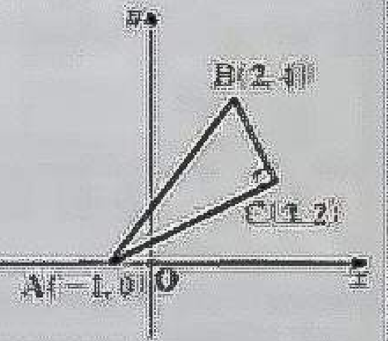
[Sol] $AB^2=(2+1)^2+(4-0)^2=25$

$BC^2=(3-2)^2+(2-4)^2=5$

$CA^2=(-1-3)^2+(0-2)^2=20$

$\therefore\ BC^2+CA^2=AB^2$

Therefore, $\triangle ABC$ is a right-angled triangle with right angle C.



1. Given $\triangle ABC$ with vertices $A(0, 3)$, $B(-3, -3)$ and $C(4, 1)$, show that $\triangle ABC$ is a right-angled triangle.

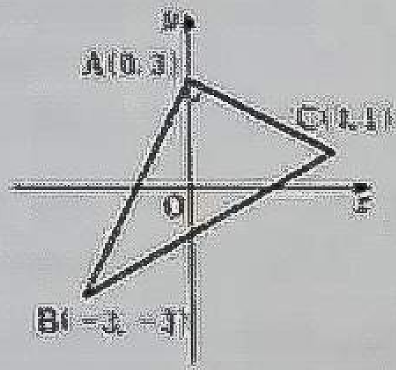
[Sol] $AB^2=(-3-0)^2+(-3-3)^2=45$

$BC^2=(4+3)^2+(1+3)^2=65$

$CA^2=(0-4)^2+(3-1)^2=20$

$\therefore\ AB^2+CA^2=BC^2$

Therefore, $\triangle ABC$ is a right-angled triangle with right angle A.



2. Given $\triangle ABC$ with vertices $A(1, 2\sqrt{3})$, $B(2, \sqrt{3})$ and $C(-1, 0)$, show that $\triangle ABC$ is a right-angled triangle.

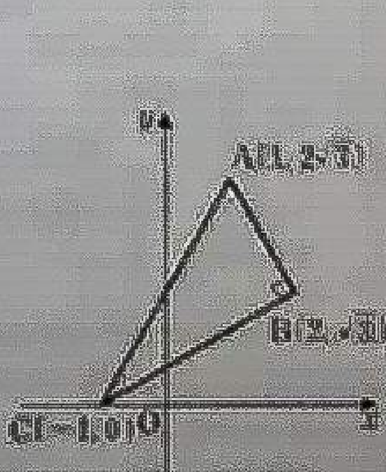
[Sol] $AB^2=(2-1)^2+(\sqrt{3}-2\sqrt{3})^2=4$

$BC^2=(-1-2)^2+(0-\sqrt{3})^2=12$

$CA^2=(1+1)^2+(2\sqrt{3}-0)^2=16$

$\therefore\ AB^2+BC^2=CA^2$

Therefore, $\triangle ABC$ is a right-angled triangle with right angle B.



M3b

3. Given $\triangle ABC$ with vertices $A(-1, 2)$, $B(2, -1)$ and $C(5, 5)$, show that $\triangle ABC$ is an isosceles triangle.

[Sol] $AB^2=(2+1)^2+(-1-2)^2=18$

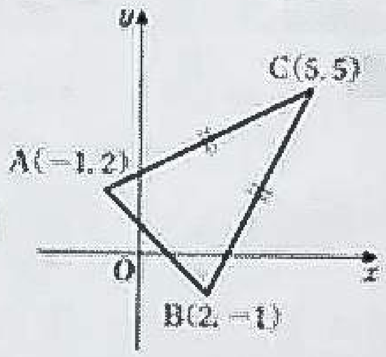
$BC^2=(5-2)^2+(5+1)^2=45$

$CA^2=(-1-5)^2+(2-5)^2=45$

$\therefore BC^2=CA^2$

$\therefore BC=CA$ Since $BC>0$ and $CA>0$

Therefore, $\triangle ABC$ is an isosceles triangle with $BC=CA$.



4. Given $\triangle ABC$ with vertices $A(5, 1)$, $B(1, 3)$ and $C(3+\sqrt{3}, 2+2\sqrt{3})$, show that $\triangle ABC$ is an equilateral triangle.

[Sol] $AB^2=(1-5)^2+(3-1)^2=20$

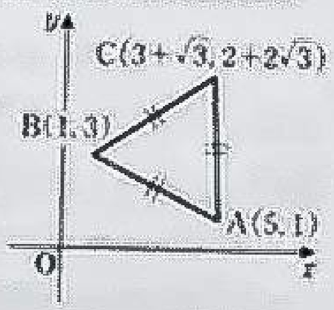
$BC^2=[(3+\sqrt{3})-1]^2+[(2+2\sqrt{3})-3]^2=20$

$CA^2=[5-(3+\sqrt{3})]^2+[1-(2+2\sqrt{3})]^2=20$

$\therefore AB^2=BC^2=CA^2$

$\therefore AB=BC=CA$

Therefore, $\triangle ABC$ is an equilateral triangle.



5. Given $\triangle ABC$ with vertices $A(3, -1)$, $B(6, 3)$ and $C(-1, 2)$, show that $\triangle ABC$ is an isosceles right-angled triangle.

[Sol] $AB^2=(6-3)^2+(3+1)^2=25$

$BC^2=(-1-6)^2+(2-3)^2=50$

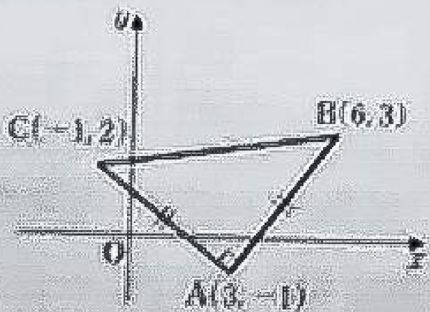
$CA^2=(3+1)^2+(-1-2)^2=25$

$\therefore AB^2+CA^2=BC^2$

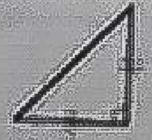
Also, $AB^2=CA^2$

So, $AB=CA$

Therefore, $\triangle ABC$ is an isosceles right-angled triangle with right angle A and $AB=CA$.



An isosceles triangle with a right angle (90°) is called an *isosceles right-angled triangle*.



Isosceles right-angled triangle

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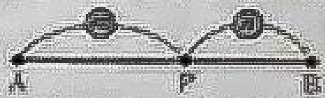
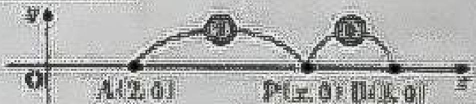
Internal Division and Internal Dividing Points

Given that point P lies on line segment AB, and AP : PB = m : n, line segment AB is said to be ***internally divided by*** P in the ratio m : n. Point P is called an ***internal dividing point***.
(m and n are positive numbers.)



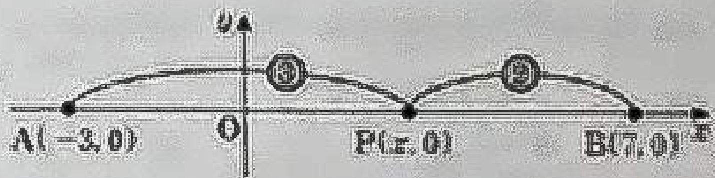
Ex. Given points A(2, 0) and B(8, 0), find the coordinates of point P(x, 0) which internally divides line segment AB in the ratio 2 : 1.

[Sol] $(x-2):(8-x)=2:1$ ← AP=x−2, PB=8−x
$2(8-x)=x-2$
$\therefore x=6 \quad \therefore$ P(6, 0)



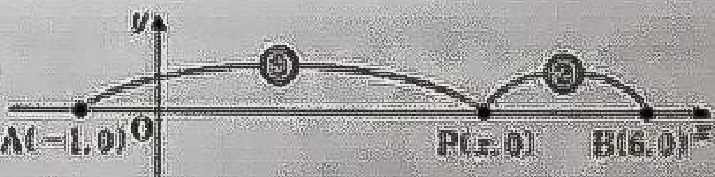
1. Given points A(−3, 0) and B(7, 0), find the coordinates of point P(x, 0) which internally divides line segment AB in the ratio 3 : 2.

[Sol] $(x+3):(7-x)=3:2$
$3(7-x)=2(x+3)$
$\therefore x=3 \quad \therefore$ P(3, 0)



2. Given points A(−1, 0) and B(6, 0), find the coordinates of point P(x, 0) which internally divides line segment AB in the ratio 5 : 2.

[Sol] $(x+1):(6-x)=5:2$
$5(6-x)=2(x+1)$
$\therefore x=4 \quad \therefore$ P(4, 0)



3. Given points A(−1, 0) and B(6, 0), find the coordinates of point P(x, 0) which internally divides line segment BA in the ratio 5 : 2. (Line segment BA is internally divided by P into line segments PB and AP.)

[Sol] $(6-x):(x+1)=5:2$
$5(x+1)=2(6-x)$
$\therefore x=1 \quad \therefore$ P(1, 0)

...en endpoint A to endpoint B on a line is called *line segment* AB.

M4b

Ex Given points $A(2, 3)$ and $B(7, 8)$, find the coordinates of point $P(x, y)$ which internally divides line segment AB in the ratio 3 : 2.

[Sol] $(x-2):(7-x)=3:2$ ⟵ Based on the x-axis

$3(7-x)=2(x-2)$

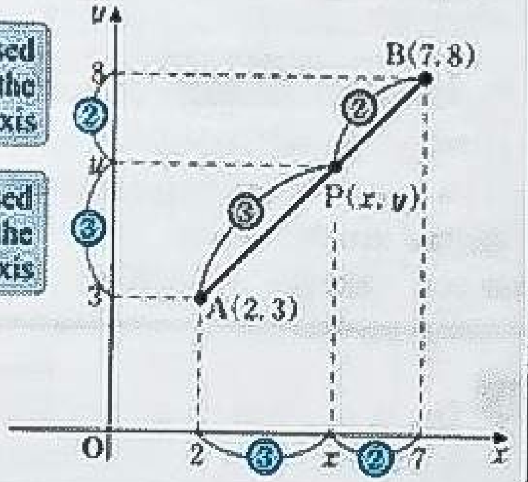
$\therefore\ x=5$

$(y-3):(8-y)=3:2$ ⟵ Based on the y-axis

$3(8-y)=2(y-3)$

$\therefore\ y=6$

$\therefore\ P(5, 6)$



1. Given points $A(-3, -1)$ and $B(3, 5)$, find the coordinates of point $P(x, y)$ which internally divides line segment AB in the ratio 2 : 1.

Sol] $(x+3):(3-x)=2:1$

$2(3-x)=x+3$

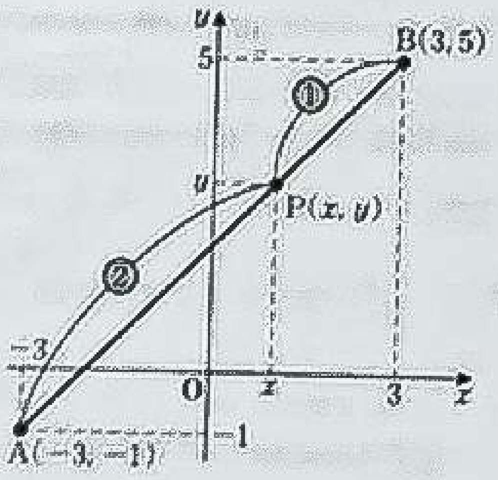
$\therefore\ x=1$

$(y+1):(5-y)=2:1$

$2(5-y)=y+1$

$\therefore\ y=3$

$\therefore\ P(1, 3)$



Given points $A(1, 2)$ and $B(5, -2)$, find the coordinates of point $P(x, y)$ which internally divides line segment AB in the ratio 3 : 1.

ol] $(x-1):(5-x)=3:1$

$3(5-x)=x-1$

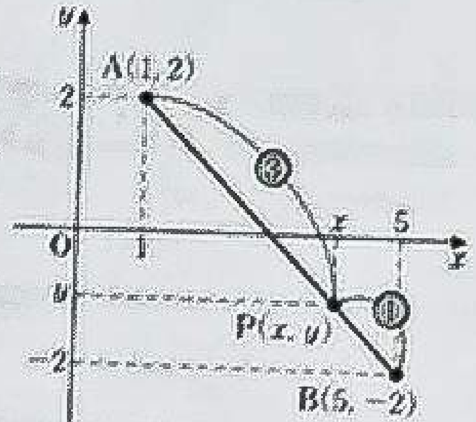
$\therefore\ x=4$

$(2-y):(y+2)=3:1$

$3(y+2)=2-y$

$\therefore\ y=-1$

$\therefore\ P(4, -1)$



M5a

KUMON

Points and Lines 1

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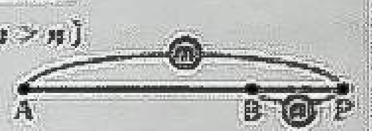
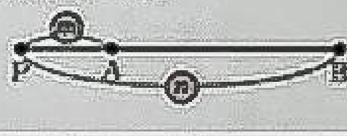
External Division and External Dividing Points

Given that point P lies on the same line as line segment AB, but outside of line segment AB, and AP : PB = $m : n$, AB is said to be ***externally divided by*** P in the ratio $m : n$. Point P is called an ***external dividing point***. (m and n are positive numbers.)

(When $m > n$)



(When $m < n$)



Ex. Given points A(3, 0) and B(7, 0), find the coordinates of point P(x, 0) which externally divides line segment AB in the ratio 3 : 1.

[Sol] $(x-3):(x-7)=3:1$ ⟵ AP $= x-3$, BP $= x-7$

$3(x-7)=x-3$

$\therefore\ x=9$

$\therefore$ P(9, 0)

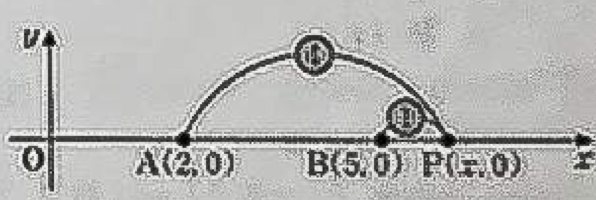
1. Given points A(2, 0) and B(5, 0), find the coordinates of point P(x, 0) which externally divides line segment AB in the ratio 4 : 1.

[Sol] $(x-2):(x-5)=4:1$

$4(x-5)=x-2$

$\therefore\ x=6$

$\therefore$ P(6, 0)



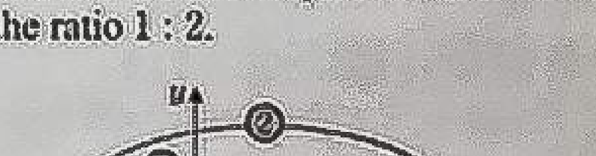
2. Given points A(1, 0) and B(4, 0), find the coordinates of point P(x, 0) which externally divides line segment AB in the ratio 1 : 2.

[Sol] $(1-x):(4-x)=1:2$

$4-x=2(1-x)$

$\therefore\ x=-2$

$\therefore$ P$(-2, 0)$



3. Given points A(2, 0) and B(4, 0), find the coordinates of point P(x, 0) which externally divides line segment AB in the ratio 3 : 2.

[Sol] $(x-2):(x-4)=3:2$

$3(x-4)=2(x-2)$

$\therefore\ x=8$

$\therefore$ P(8, 0)

M5b

Ex Given points A(1, 2) and B(5, 6), find the coordinates of point P(x, y) which externally divides line segment AB in the ratio 3 : 1.

[Sol] $(x-1):(x-5)=3:1$

$3(x-5)=x-1$

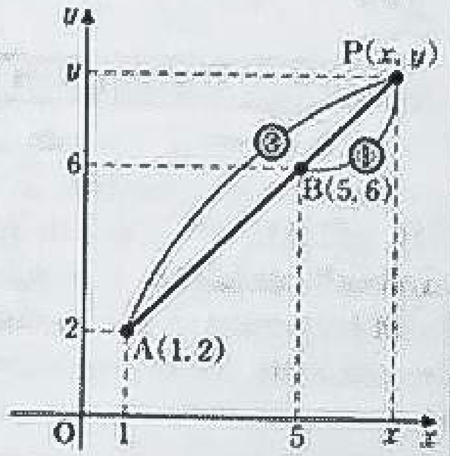
$\therefore x=7$

$(y-2):(y-6)=3:1$

$3(y-6)=y-2$

$\therefore y=8$

$\therefore$ P(7, 8)



1. Given points A(−1, 2) and B(5, 4), find the coordinates of point P(x, y) which externally divides line segment AB in the ratio 2 : 1.

[Sol] $(x+1):(x-5)=2:1$

$2(x-5)=x+1$

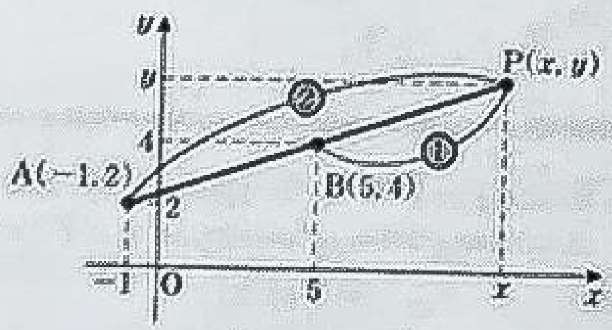
$\therefore x=11$

$(y-2):(y-4)=2:1$

$2(y-4)=y-2$

$\therefore y=6$

$\therefore$ P(11, 6)



Given points A(−2, 3) and B(1, 2), find the coordinates of point P(x, y) which externally divides line segment AB in the ratio 2 : 3.

[Sol] $(-2-x):(1-x)=2:3$

$2(1-x)=3(-2-x)$

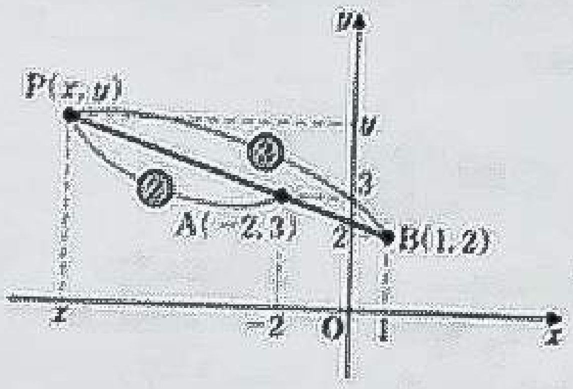
$\therefore x=-8$

$(y-3):(y-2)=2:3$

$2(y-2)=3(y-3)$

$\therefore y=5$

$\therefore$ P(−8, 5)



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Let the coordinates of two points A and B be $A(x_1, y_1)$ and $B(x_2, y_2)$ as shown in the diagram $(x_2 > x_1)$. Find the coordinates of point $P(x, y)$ which internally divides line segment AB in the ratio $m : n$.

[Sol] $(x-x_1):(x_2-x)=m:n$

$$m(x_2-x)=n(x-x_1)$$

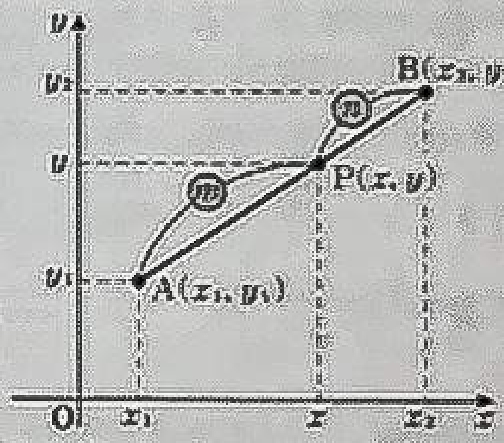
$$\therefore\ x=\boxed{\frac{nx_1+mx_2}{m+n}}$$

$$(y-y_1):(y_2-y)=m:n$$

$$m(y_2-y)=n(y-y_1)$$

$$\therefore\ y=\boxed{\frac{ny_1+my_2}{m+n}}$$

$$\therefore\ P\left(\boxed{\frac{nx_1+mx_2}{m+n}},\ \boxed{\frac{ny_1+my_2}{m+n}}\right)$$



Answers: $\frac{nx_1+mx_2}{m+n}$, $\frac{ny_1+my_2}{m+n}$, $\frac{nx_1+mx_2}{m+n}$, $\frac{ny_1+my_2}{m+n}$

Similarly, find the coordinates of point $Q(x, y)$ which externally divides line segment AB in the ratio $m : n$. When $m > n$, $x > x_2 > x_1$ is true.

$$(x-x_1):(x-x_2)=m:n$$

$$m(x-x_2)=n(x-x_1)$$

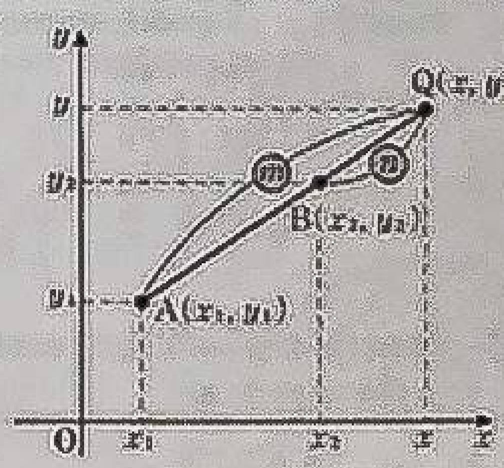
$$\therefore\ x=\frac{-nx_1+mx_2}{m-n}$$

$$(y-y_1):(y-y_2)=m:n$$

$$m(y-y_2)=n(y-y_1)$$

$$\therefore\ y=\frac{-ny_1+my_2}{m-n}$$

$$\therefore\ Q\left(\frac{-nx_1+mx_2}{m-n},\ \frac{-ny_1+my_2}{m-n}\right)$$



n within the coordinates of the point which internally divides the line segment are all replaced by $-n$.

This is also true when $m < n$.

M6b

Coordinates of Internal/External Dividing Points

Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, the coordinates of the points that divide line segment AB in the ratio $m : n$ are as follows:

Internally, $\left(\frac{nx_1+mx_2}{m+n}, \frac{ny_1+my_2}{m+n}\right)$

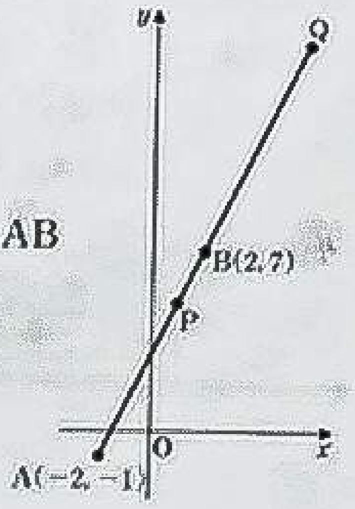
Externally, $\left(\frac{-nx_1+mx_2}{m-n}, \frac{-ny_1+my_2}{m-n}\right)$

1. Given points $A(-2, -1)$ and $B(2, 7)$, find the coordinates of the following points by using the formulas above.

(1) Point $P(x, y)$ which internally divides line segment AB in the ratio 3 : 1.

[Sol] $x=\frac{1\cdot(-2)+3\cdot 2}{3+1}=1$

$y=\frac{1\cdot(-1)+3\cdot 7}{3+1}=5$ $\quad\therefore$ **P(1, 5)**



(2) Point $Q(x, y)$ which externally divides line segment AB in the ratio 2 : 1.

[Sol] $x=\frac{-1\cdot(-2)+2\cdot 2}{2-1}=6$

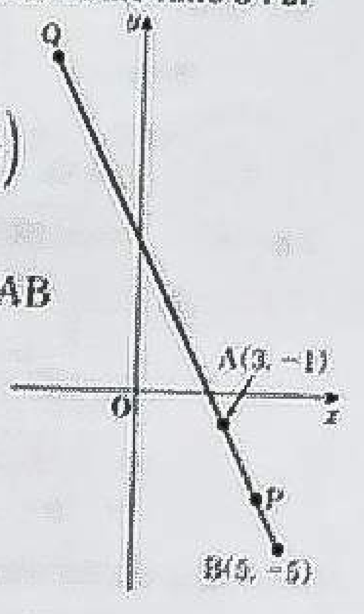
$y=\frac{-1\cdot(-1)+2\cdot 7}{2-1}=15$ $\quad\therefore$ **Q(6, 15)**

2. Given points $A(3, -1)$ and $B(5, -5)$, find the coordinates of the following points by using the formulas above.

(1) Point $P(x, y)$ which internally divides line segment AB in the ratio 3 : 2.

[Sol] $x=\frac{2\cdot 3+3\cdot 5}{3+2}=\frac{21}{5}$

$y=\frac{2\cdot(-1)+3\cdot(-5)}{3+2}=-\frac{17}{5}$ $\quad\therefore$ $\mathbf{P}\left(\frac{21}{5}, -\frac{17}{5}\right)$



(2) Point $Q(x, y)$ which externally divides line segment AB in the ratio 3 : 4.

[Sol] $x=\frac{-4\cdot 3+3\cdot 5}{3-4}=-3$

$y=\frac{-4\cdot(-1)+3\cdot(-5)}{3-4}=11$ $\quad\therefore$ **Q(−3, 11)**

Points and Lines 1

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Date / /

Time : to :

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1. Let the coordinates of A and B be $A(x_1, y_1)$ and $B(x_2, y_2)$. Find the coordinates of point $P(x, y)$ which internally divides line segment AB in the ratio 1 : 1. (Express point P in terms of x_1, x_2, y_1 and y_2.)

[Sol] $x = \dfrac{1 \cdot x_1 + 1 \cdot x_2}{1+1}$

$= \dfrac{x_1 + x_2}{2}$

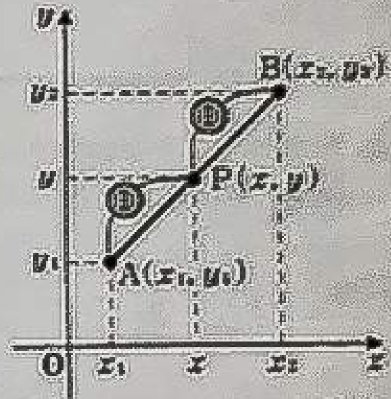
$y = \dfrac{1 \cdot y_1 + 1 \cdot y_2}{1+1}$

$= \dfrac{y_1 + y_2}{2}$

$\therefore P\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

The coordinates of the point internally dividing the line in the ratio $m : n$ are

$$\left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$$



A point which internally divides line segment AB in the ratio 1 : 1 is called the ***midpoint*** of the line segment. From the above, the following formula is true in general.

Midpoint

The coordinates of the midpoint M of line segment AB, where $A(x_1, y_1)$ and $B(x_2, y_2)$, are given by

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

2. Find the midpoint of the line segment formed by the given points.

Ex. A(2, 1), B(4, 7)

[Sol] $\left(\dfrac{2+4}{2}, \dfrac{1+7}{2}\right)$

$\therefore (3, 4)$

(2) A(3, −7), B(−5, 3)

[Sol] $\left(\dfrac{3-5}{2}, \dfrac{-7+3}{2}\right)$

$\therefore (-1, -2)$

(1) A(4, 0), B(6, 2)

[Sol] $\left(\dfrac{4+6}{2}, \dfrac{0+2}{2}\right)$

$\therefore (5, 1)$

(3) A(−1, −5), B(3, −1)

[Sol] $\left(\dfrac{-1+3}{2}, \dfrac{-5-1}{2}\right)$

$\therefore (1, -3)$

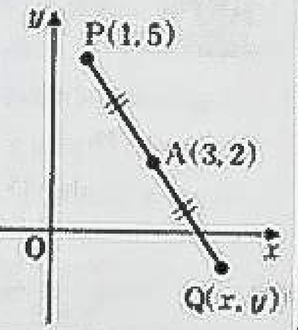
M7b

Ex Find the coordinates of point $Q(x, y)$ which is symmetric to point $P(1, 5)$ with respect to point $A(3, 2)$.

[Sol] $3=\dfrac{1+x}{2}$, $2=\dfrac{5+y}{2}$ ⟵ Since point A is the midpoint of line segment PQ

$\therefore x=5,\ y=-1$

$\therefore Q(5, -1)$

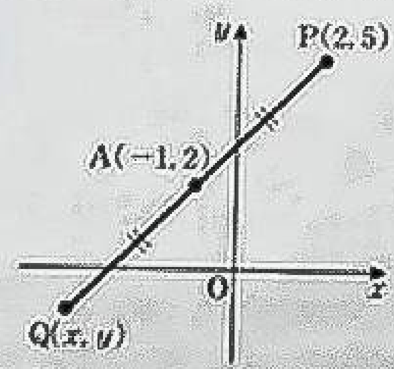


3. Find the coordinates of point $Q(x, y)$ which is symmetric to point $P(2, 5)$ with respect to point $A(-1, 2)$.

[Sol] $-1=\dfrac{2+x}{2}$, $2=\dfrac{5+y}{2}$

$\therefore x=-4,\ y=-1$

$\therefore Q(-4, -1)$

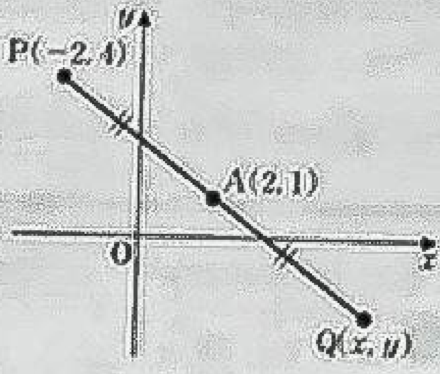


. Find the coordinates of point $Q(x, y)$ which is symmetric to point $P(-2, 4)$ with respect to point $A(2, 1)$.

[Sol] $2=\dfrac{-2+x}{2}$, $1=\dfrac{4+y}{2}$

$\therefore x=6,\ y=-2$

$\therefore Q(6, -2)$

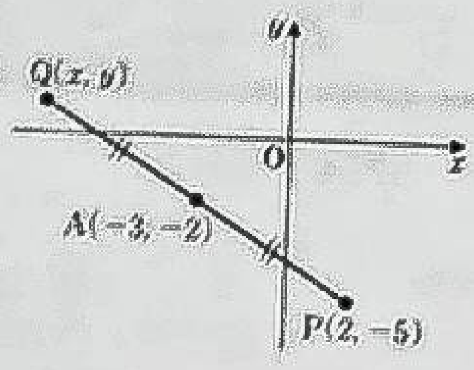


Find the coordinates of point $Q(x, y)$ which is symmetric to point $P(2, -5)$ with respect to point $A(-3, -2)$.

[Sol] $-3=\dfrac{2+x}{2}$, $-2=\dfrac{-5+y}{2}$

$\therefore x=-8,\ y=1$

$\therefore Q(-8, 1)$



Points and Lines 1

Name

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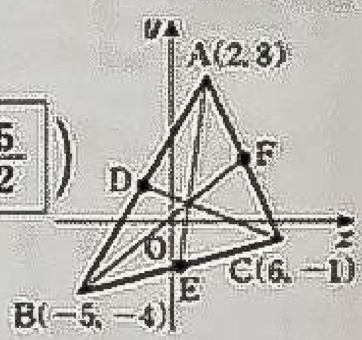
Time : to :

1. Given $\triangle ABC$ with vertices $A(2, 8)$, $B(-5, -4)$ and $C(6, -1)$, solve the following questions.

(1) Points D, E and F are the midpoints of sides AB, BC and CA respectively. Find their coordinates.



[Sol] $D:\left(\frac{2-5}{2}, \frac{8-4}{2}\right)$ $\therefore D\left(\boxed{-\frac{3}{2}}, \boxed{2}\right)$

$E:\left(\boxed{\frac{-5+6}{2}}, \boxed{\frac{-4-1}{2}}\right)$ $\therefore E\left(\boxed{\frac{1}{2}}, \boxed{-\frac{5}{2}}\right)$

$F:\left(\boxed{\frac{6+2}{2}}, \boxed{\frac{-1+8}{2}}\right)$ $\therefore F\left(\boxed{4}, \boxed{\frac{7}{2}}\right)$

(2) Find the coordinates of point $G_1(x, y)$ which internally divides line segment AE in the ratio $2:1$.

[Sol] $x=\frac{1\cdot 2+2\cdot\frac{1}{2}}{2+1}=1$, $y=\frac{1\cdot 8+2\cdot\left(-\frac{5}{2}\right)}{2+1}=1$ ← $A(2, 8)$, $E\left(\frac{1}{2}, -\frac{5}{2}\right)$

$\therefore G_1(1, 1)$

(3) Find the coordinates of point $G_2(x, y)$ which internally divides line segment BF in the ratio $2:1$.

[Sol] $x=\frac{1\cdot(-5)+2\cdot 4}{2+1}=1$, $y=\frac{1\cdot(-4)+2\cdot\frac{7}{2}}{2+1}=1$ ← $B(-5, -4)$, $F\left(4, \frac{7}{2}\right)$

$\therefore G_2(1, 1)$

(4) Find the coordinates of point $G_3(x, y)$ which internally divides line segment CD in the ratio $2:1$.

[Sol] $x=\frac{1\cdot 6+2\cdot\left(-\frac{3}{2}\right)}{2+1}=1$, $y=\frac{1\cdot(-1)+2\cdot 2}{2+1}=1$ ← $C(6, -1)$, $D\left(-\frac{3}{2}, 2\right)$

$\therefore G_3(1, 1)$

The line segment connecting a vertex of a triangle to the midpoint of the opposite side is called a ***median***. The three medians of a triangle intersect at one point which internally divides each median in the ratio $2:1$. The point at which the three medians intersect is the ***center of gravity*** of the triangle.

M8b

Center of Gravity of Triangles

Given $\triangle ABC$ with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, the coordinates of the center of gravity G are given by

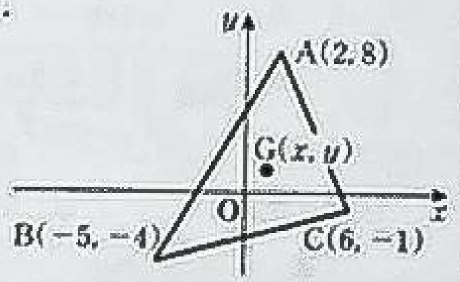
$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Ex Given $\triangle ABC$ with vertices $A(2, 8)$, $B(-5, -4)$ and $C(6, -1)$, find the coordinates of the center of gravity $G(x, y)$.

[Sol] $x=\dfrac{2-5+6}{3}=1$

$y=\dfrac{8-4-1}{3}=1$

$\therefore G(1, 1)$

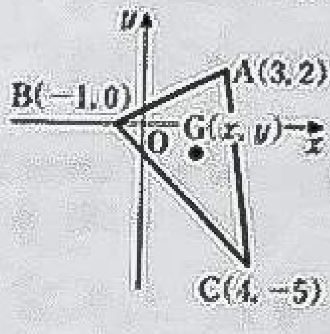


. Given $\triangle ABC$ with vertices $A(3, 2)$, $B(-1, 0)$ and $C(4, -5)$, find the coordinates of the center of gravity $G(x, y)$.

ol] $x=\dfrac{3-1+4}{3}=2$

$y=\dfrac{2+0-5}{3}=-1$

$\therefore G(2, -1)$

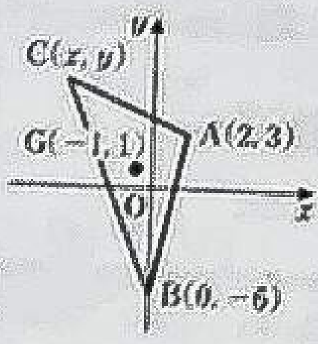


Given $\triangle ABC$ with vertices $A(2, 3)$ and $B(0, -6)$ and center of gravity $G(-1, 1)$, find the coordinates of the other vertex $C(x, y)$.

ol] $-1=\dfrac{2+0+x}{3}$, $1=\dfrac{3-6+y}{3}$

$\therefore x=-5$, $y=6$

$\therefore C(-5, 6)$



Points and Lines 1

Name

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Time : to :

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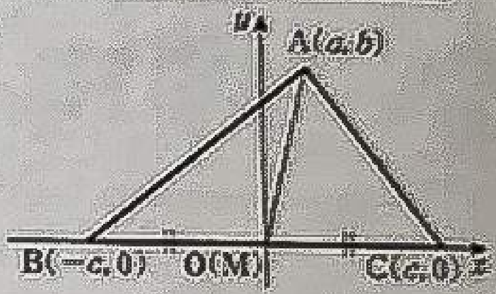
1. Given $\triangle$ABC with M as the midpoint of side BC, prove the identity
   $AB^2+AC^2=2(AM^2+BM^2)$

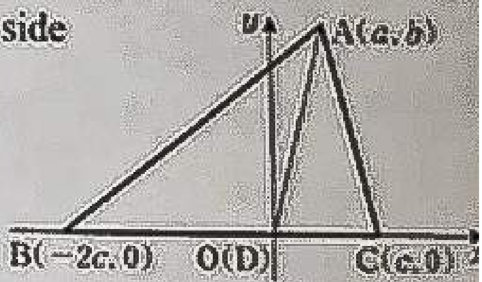


[Sol] Placing side BC on the x-axis and M at the origin, let points be A(a, b), B$(-c, 0)$ and C$(c, 0)$.

$$AB^2+AC^2=[(-c-a)^2+(0-b)^2]+[\boxed{(c-a)^2+(0-b)^2}]$$
$$=\boxed{2a^2+2b^2+2c^2}\ \cdots①$$
$$2(AM^2+BM^2)=2[\boxed{(a^2+b^2)+c^2}]$$
$$=\boxed{2a^2+2b^2+2c^2}\ \cdots②$$

From ① and ②, $AB^2+AC^2=2(AM^2+BM^2)$

2. Given $\triangle$ABC where point D internally divides side BC in the ratio 2 : 1, prove the identity
   $AB^2+2AC^2=3AD^2+6CD^2$



[Sol] Placing side BC on the x-axis and D at the origin, let points be A(a, b), B$(-2c, 0)$ and C$(c, 0)$.

$$AB^2+2AC^2=[(-2c-a)^2+(0-b)^2]+2[(c-a)^2+(0-b)^2]$$
$$=3a^2+3b^2+6c^2\ \cdots①$$
$$3AD^2+6CD^2=3(a^2+b^2)+6c^2$$
$$=3a^2+3b^2+6c^2\ \cdots②$$

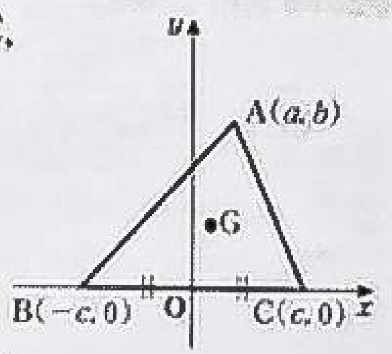
From ① and ②, $AB^2+2AC^2=3AD^2+6CD^2$

$AB^2+AC^2=2(AM^2+BM^2)$, which is proved in question 1, is called the *Parallelogram Theo*

M9b

3. Given that point G is the center of gravity of $\triangle ABC$, prove the identity

$$AG^2+BG^2+CG^2=\frac{1}{3}(AB^2+BC^2+CA^2)$$



[Sol] Placing side BC on the x-axis, let points be A(a, b), B$(-c, 0)$ and C$(c, 0)$.

Since the coordinates of G are $\left(\boxed{\frac{a}{3}}, \boxed{\frac{b}{3}}\right)$,

Let G be (x, y).
$x=\frac{a-c+c}{3}$, $y=\frac{b+0+0}{3}$

$$AG^2=\left(\frac{a}{3}-a\right)^2+\left(\frac{b}{3}-b\right)^2=\frac{4}{9}a^2+\frac{4}{9}b^2$$

$$BG^2=\left(\frac{a}{3}+c\right)^2+\left(\frac{b}{3}-0\right)^2=\frac{a^2}{9}+\frac{2}{3}ac+c^2+\frac{b^2}{9}$$

$$CG^2=\left(\frac{a}{3}-c\right)^2+\left(\frac{b}{3}-0\right)^2=\frac{a^2}{9}-\frac{2}{3}ac+c^2+\frac{b^2}{9}$$

Therefore,

$$AG^2+BG^2+CG^2=\frac{2}{3}a^2+\frac{2}{3}b^2+2c^2 \quad \cdots ①$$

Also,

$$AB^2=(-c-a)^2+(0-b)^2=a^2+2ac+c^2+b^2$$

$$BC^2=(c+c)^2=4c^2$$

$$CA^2=(a-c)^2+(b-0)^2=a^2-2ac+c^2+b^2$$

Therefore,

$$\frac{1}{3}(AB^2+BC^2+CA^2)=\frac{2}{3}a^2+\frac{2}{3}b^2+2c^2 \quad \cdots ②$$

From ① and ②,

$$AG^2+BG^2+CG^2=\frac{1}{3}(AB^2+BC^2+CA^2)$$

Points and Lines 1

Name

Date / /

Time : to :

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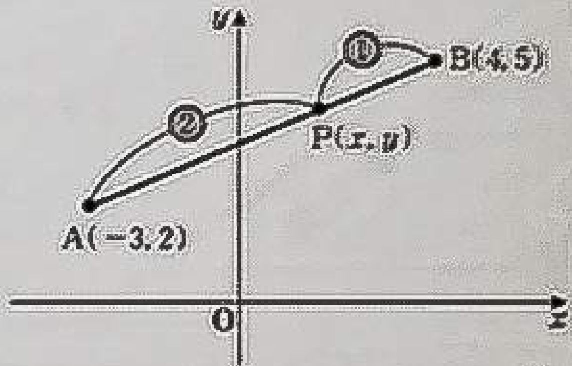
1. Given points A$(-3, 2)$ and B$(4, 5)$, find the coordinates of point P(x, y) which internally divides line segment AB in the ratio $2:1$. ➡ M6

[Sol] $x=\dfrac{1\cdot(-3)+2\cdot4}{2+1}=\dfrac{5}{3}$

$y=\dfrac{1\cdot2+2\cdot5}{2+1}=4$

$\therefore$ P$\left(\dfrac{5}{3}, 4\right)$



Alternative Solution

$(x+3):(4-x)=2:1$

$2(4-x)=x+3$

$\therefore x=\dfrac{5}{3}$

$(y-2):(5-y)=2:1$

$2(5-y)=y-2$

$\therefore y=4$

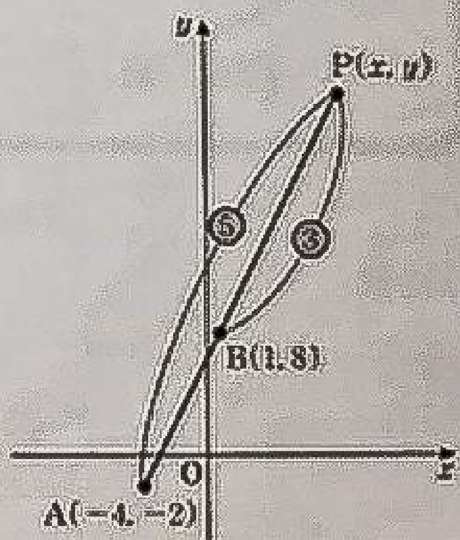
$\therefore$ P$\left(\dfrac{5}{3}, 4\right)$

2. Given points A$(-4, -2)$ and B$(1, 8)$, find the coordinates of point P(x, y) which externally divides line segment AB in the ratio $5:3$. ➡ M6

[Sol] $x=\dfrac{-3\cdot(-4)+5\cdot1}{5-3}=\dfrac{17}{2}$

$y=\dfrac{-3\cdot(-2)+5\cdot8}{5-3}=23$

$\therefore$ P$\left(\dfrac{17}{2}, 23\right)$



Alternative Solution

$(x+4):(x-1)=5:3$

$5(x-1)=3(x+4)$

$\therefore x=\dfrac{17}{2}$

$(y+2):(y-8)=5:3$

$5(y-8)=3(y+2)$

$\therefore y=23$

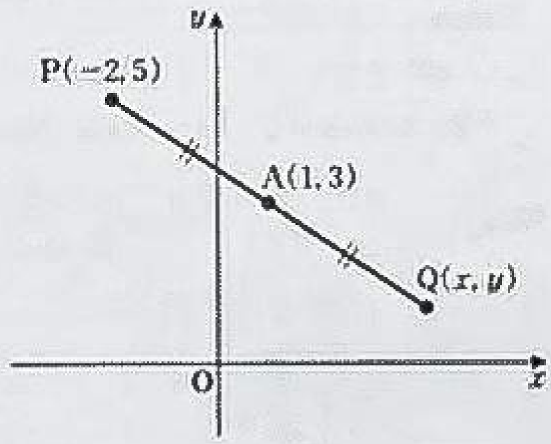
$\therefore$ P$\left(\dfrac{17}{2}, 23\right)$

3. Find the coordinates of point $Q(x, y)$ which is symmetric to point $P(-2, 5)$ with respect to point $A(1, 3)$. ➡ M7

[Sol] $1=\frac{-2+x}{2}$, $3=\frac{5+y}{2}$

$\therefore x=4,\ y=1$

$\therefore Q(4, 1)$

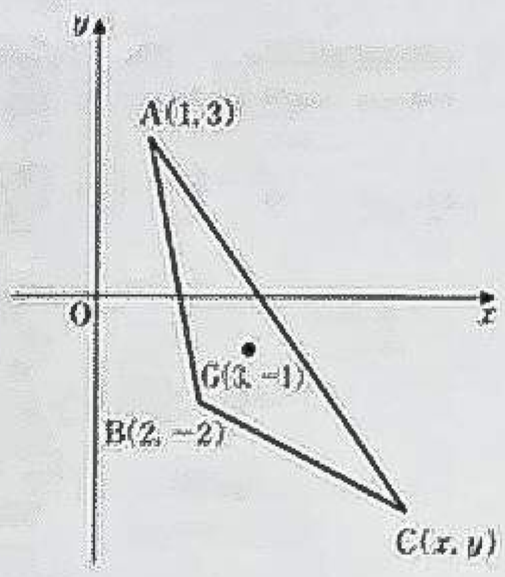


4. Given $\triangle ABC$ with vertices $A(1, 3)$ and $B(2, -2)$ and center of gravity $G(3, -1)$, find the coordinates of the other vertex $C(x, y)$. ➡ M8

[Sol] $3=\frac{1+2+x}{3}$, $-1=\frac{3-2+y}{3}$

$\therefore x=6,\ y=-4$

$\therefore C(6, -4)$



Points and Lines 2

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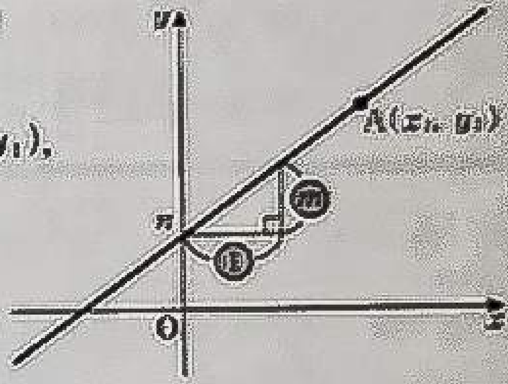
Find the equation of the line passing through point $A(x_1, y_1)$ with slope m.



[Sol] The equation of a line with slope m is

$y=mx+n$ ⋯①

Since line ① passes through point $A(x_1, y_1)$,

$y_1=\boxed{mx_1+n}$ ⋯②

From ① and ②,

$y-y_1=\boxed{m(x-x_1)}$

Answers: mx_1+n, $m(x-x_1)$

Equation of a Line I

The equation of a line passing through point (x_1, y_1) with slope m is

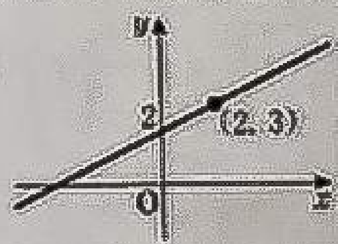
$y-y_1=m(x-x_1)$

1. Find the equation of each given line.

Ex. The line passing through point $(2, 3)$ with slope $\frac{1}{2}$.

[Sol] $y-3=\frac{1}{2}(x-2)$

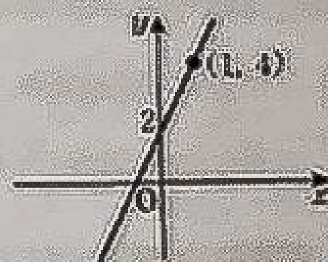
$\therefore\ y=\frac{1}{2}x+2$



(1) The line passing through point $(1, 4)$ with slope 2.

[Sol] $y-4=2(x-1)$

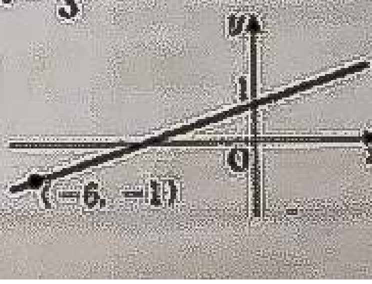
$\therefore\ y=2x+2$



(2) The line passing through point $(-6, -1)$ with slope $\frac{1}{3}$.

[Sol] $y+1=\frac{1}{3}(x+6)$

$\therefore\ y=\frac{1}{3}x+1$



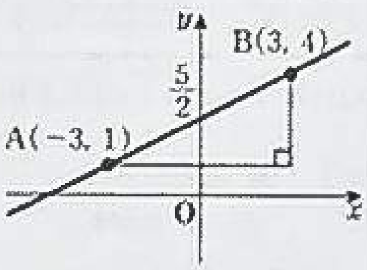
2. Find the equation of the line passing through the two given points A and B.

$A(-3, 1)$, $B(3, 4)$

[Sol] The slope of line AB is $\frac{4-1}{3+3}=\frac{1}{2}$.

$y-1=\frac{1}{2}(x+3)$ ← Passing through point $A(-3, 1)$

$\therefore\ y=\frac{1}{2}x+\frac{5}{2}$

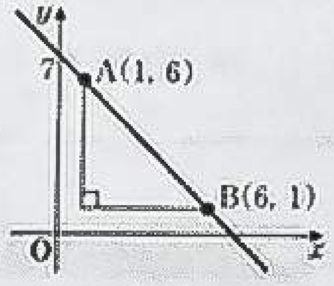


(1) $A(1, 6)$, $B(6, 1)$

[Sol] The slope of line AB is $\frac{1-6}{6-1}=-1$.

$y-6=-1\cdot(x-1)$

$\therefore\ y=-x+7$

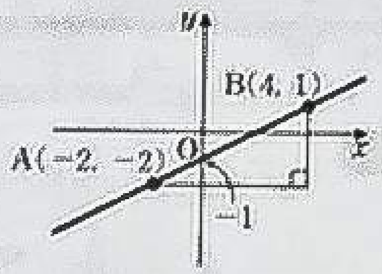


(2) $A(-2, -2)$, $B(4, 1)$

[Sol] The slope of line AB is $\frac{1+2}{4+2}=\frac{1}{2}$.

$y+2=\frac{1}{2}(x+2)$

$\therefore\ y=\frac{1}{2}x-1$

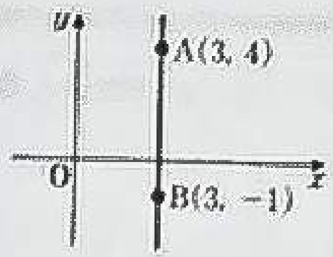


(3) $A(3, 4)$, $B(3, -1)$

[Sol] The x-coordinates of both points are $\boxed{3}$.

From the graph,

the equation of line AB is $x=\boxed{3}$.



Equation of a Line II

The equation of a line passing through points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$, when $x_1 \neq x_2$

$x=x_1$, when $x_1=x_2$

Points and Lines 2

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Find a for which all three given points lie on the same line.

Ex. A$(3, -1)$, B$(-2, 9)$, C$(a, -5)$

[Sol] The equation of line AB is

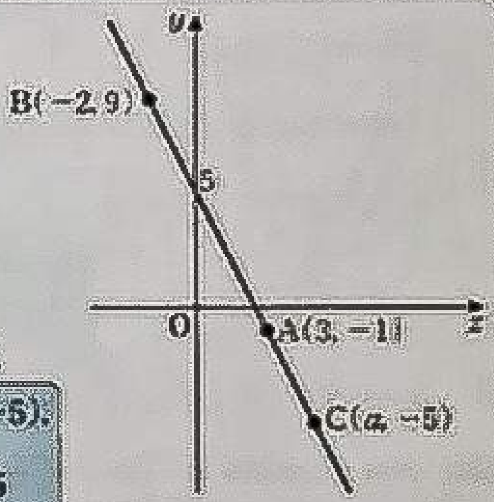
$$y+1=\frac{9+1}{-2-3}(x-3)$$

So, $y=-2x+5$ ⋯①

Since line ① passes through point C,

$-5=-2a+5$ ⟵ Since C$(a, -5)$, substitute $x=a$, $y=-5$ into ①.

$\therefore a=5$



(1) A$(2, 3)$, B$(4, 9)$, C$(5, a)$

[Sol] The equation of line AB is

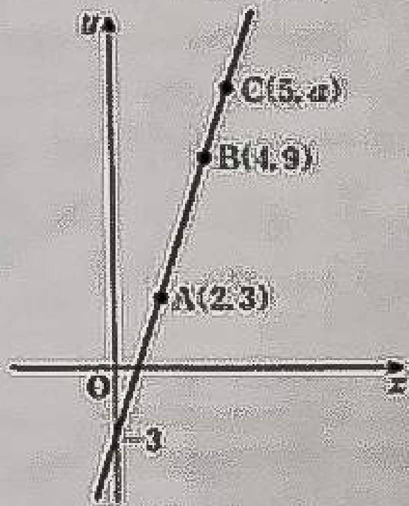
$$y-3=\frac{9-3}{4-2}(x-2)$$

So, $y=3x-3$ ⋯①

Since line ① passes through point C,

$a=3\cdot5-3$

$\therefore a=12$



(2) A$(1, 5)$, B$(2, 7)$, C(a, a)

[Sol] The equation of line AB is

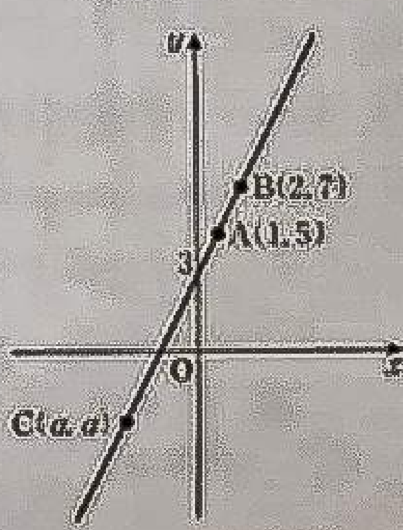
$$y-5=\frac{7-5}{2-1}(x-1)$$

So, $y=2x+3$ ⋯①

Since line ① passes through point C,

$a=2a+3$

$\therefore a=-3$



M12b

(3) A(−2, 3), B(1, 2), C(a, $a+9$)

[Sol] The equation of line AB is

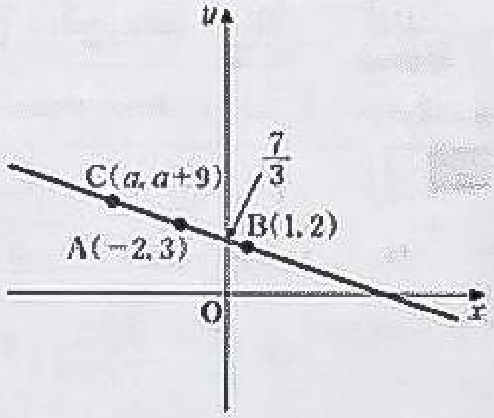
$$y-3=\frac{2-3}{1+2}(x+2)$$

So, $y=-\frac{1}{3}x+\frac{7}{3}$ ⋯①

Since line ① passes through point C,

$$a+9=-\frac{1}{3}a+\frac{7}{3}$$

$\therefore\ a=-5$



(4) A(0, 8), B(4, 0), C(a, a^2)

[Sol] The equation of line AB is

$$y-8=\frac{0-8}{4-0}(x-0)$$

So, $y=-2x+8$ ⋯①

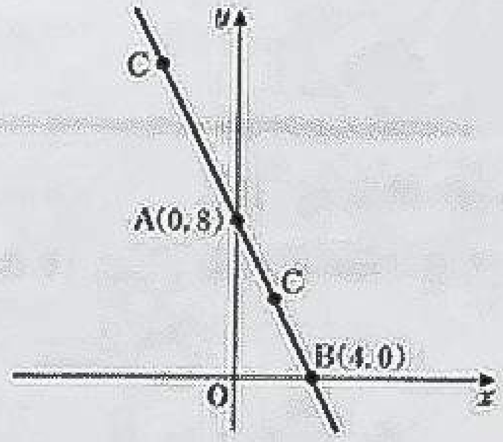
Since line ① passes through point C,

$a^2=-2a+8$

$a^2+2a-8=0$

$(a+4)(a-2)=0$

$\therefore\ a=-4,\ 2$



(5) A(2, 5), B(0, a), C(a, 3)

[Sol] The equation of line AB is

$$y-5=\frac{a-5}{0-2}(x-2)$$

So, $y=-\frac{1}{2}(a-5)x+a$ ⋯①

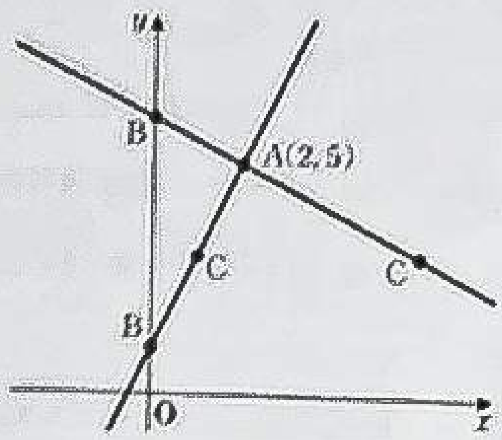
Since line ① passes through point C,

$3=-\frac{1}{2}(a-5)a+a$

$a^2-7a+6=0$

$(a-1)(a-6)=0$

$\therefore\ a=1,\ 6$



Points and Lines 2

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1. Find the equation of the line passing through the point of intersection of two lines $2x-3y=-1$ ···① and $2x-y=1$ ···② and point $(2, -2)$.

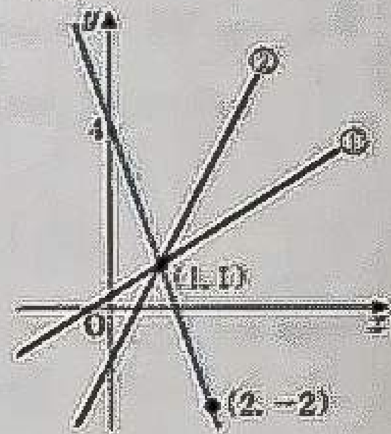
[Sol] From ① and ②, $x=1$, $y=1$

Therefore, the point of intersection is $(1, 1)$.

Thus, the equation of the line is

$$y-1=\frac{-2-1}{2-1}(x-1) \longleftarrow$$ Passing through points $(1, 1)$ and $(2, -2)$

$\therefore\ y=-3x+4 \quad [3x+y=4]$



2. Find the equation of the line passing through the point of intersection of two lines $2x-3y=5$ ···① and $x-5y=-1$ ···② and point $(6, 5)$.

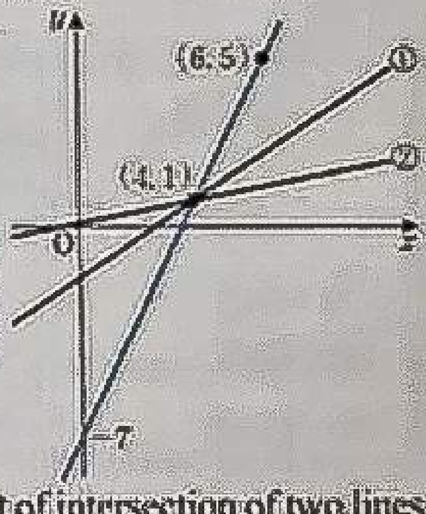
[Sol] From ① and ②, $x=4$, $y=1$

Therefore, the point of intersection is $(4, 1)$.

Thus, the equation of the line is

$$y-1=\frac{5-1}{6-4}(x-4) \longleftarrow$$ Passing through points $(4, 1)$ and $(6, 5)$

$\therefore\ y=2x-7 \quad [2x-y=7]$



3. Find the equation of the line passing through the point of intersection of two lines $3x+4y-8=0$ ···① and $6x-5y-3=0$ ···② and point $(2, -1)$.

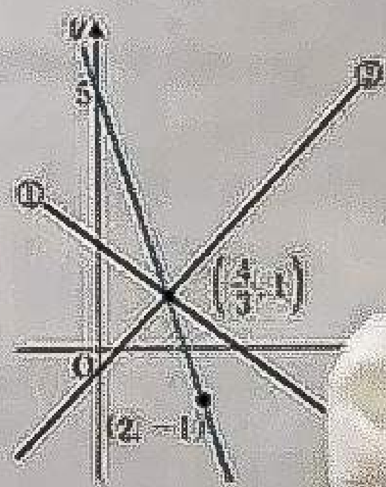
[Sol] From ① and ②, $x=\frac{4}{3}$, $y=1$

Therefore, the point of intersection is $\left(\frac{4}{3}, 1\right)$.

Thus, the equation of the line is

$$y-1=\frac{-1-1}{2-\frac{4}{3}}\left(x-\frac{4}{3}\right) \longleftarrow$$ Passing through points $\left(\frac{4}{3}, 1\right)$ and $(2, -1)$

$\therefore\ y=-3x+5 \quad [3x+y-5=0]$



M13b

4. Find the constant m for which all three lines $2x-y-4=0$ ⋯①, $x+2y+3=0$ ⋯② and $mx+y-1=0$ ⋯③ intersect at one point.

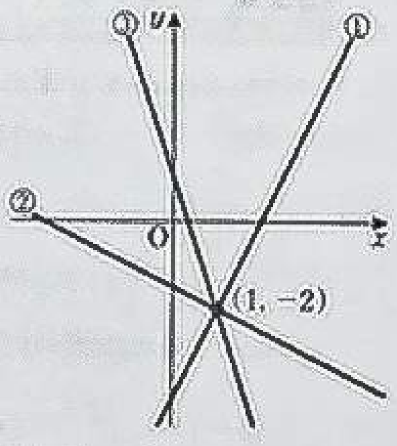
[Sol] From ① and ②, $x=1$, $y=-2$

Therefore, the point of intersection is $(1, -2)$.

Since line ③ passes through point $(1, -2)$,

$m\cdot1-2-1=0$ ← Substituting $x=1$, $y=-2$ into ③

$\therefore\ m=3$



5. Find the constant m for which all three lines $x-2y+8=0$ ⋯①, $2x+3y-5=0$ ⋯② and $mx-y-3m+4=0$ ⋯③ intersect at one point.

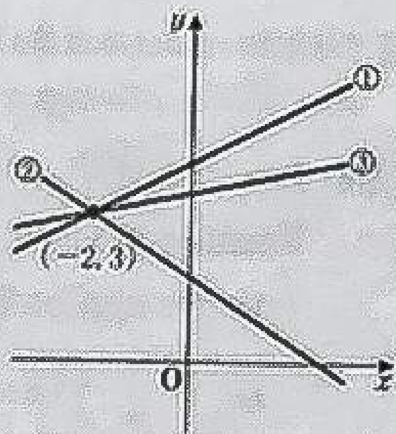
[Sol] From ① and ②, $x=-2$, $y=3$

Therefore, the point of intersection is $(-2, 3)$.

Since line ③ passes through point $(-2, 3)$,

$m\cdot(-2)-3-3m+4=0$

$\therefore\ m=\dfrac{1}{5}$



6. Find the constant m for which all three lines $x-y=-m$ ⋯①, $2x+y=m-1$ ⋯② and $x+5y=4m+1$ ⋯③ intersect at one point.

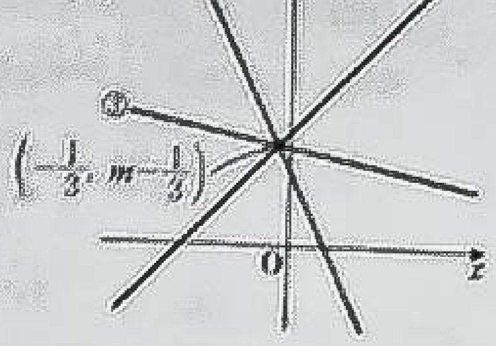
[Sol] From ① and ②, $x=-\dfrac{1}{3}$, $y=m-\dfrac{1}{3}$

Therefore, the point of intersection is $\left(-\dfrac{1}{3}, m-\dfrac{1}{3}\right)$.

Since line ③ passes through point $\left(-\dfrac{1}{3}, m-\dfrac{1}{3}\right)$,

$-\dfrac{1}{3}+5\left(m-\dfrac{1}{3}\right)=4m+1$

$\therefore\ m=3$



Points and Lines 2

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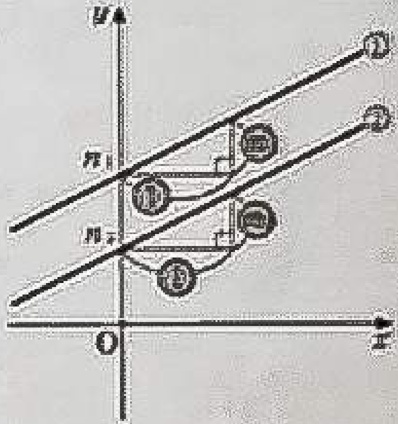
Two lines

$y=m_1x+n_1$ ···①

$y=m_2x+n_2$ ···②

are parallel if their slopes are equal.



Therefore, the following is true.

Parallel Condition

For two lines $y=m_1x+n_1$ and $y=m_2x+n_2$,
the parallel condition is $m_1=m_2$.

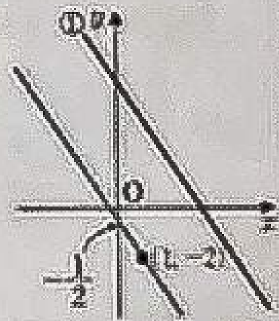
Ex. Find the equation of the line which passes through point $(1, -2)$ and is parallel to line $3x+2y-10=0$ ···①.

[Sol] The slope of line ① is $-\frac{3}{2}$. ⟵ Rearranging ①, $y=-\frac{3}{2}x+5$

Therefore, the equation of the line is

$y+2=-\frac{3}{2}(x-1)$ ⟵ Passing through point $(1, -2)$

$\therefore\ 3x+2y+1=0$



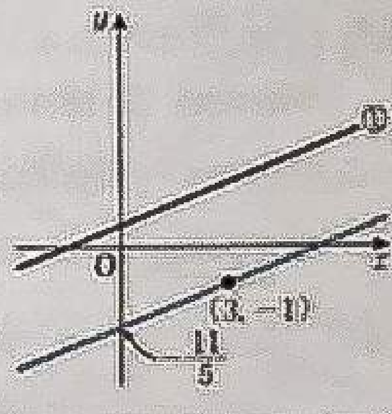
1. Find the equation of the line which passes through point $(3, -1)$ and is parallel to line $2x-5y+3=0$ ···①.

[Sol] The slope of line ① is $\frac{2}{5}$.

Therefore, the equation of the line is

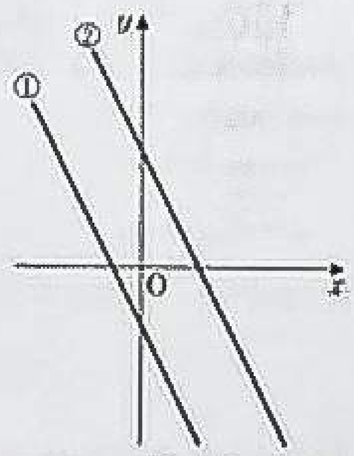
$y+1=\frac{2}{5}(x-3)$

$\therefore\ 2x-5y-11=0$ $\left[y=\frac{2}{5}x-\frac{11}{5}\right]$



Two lines $y=m_1x+n_1$ and $y=m_2x+n_2$ overlap when $m_1=m_2$ and $n_1=n_2$.
These two lines are considered to be parallel as well.

2. Find the constant a for which two lines $ax-4y=6$ …① and $2x+y=3$ …② are parallel.



[Sol] The slope of line ① is $\boxed{\frac{a}{4}}$.

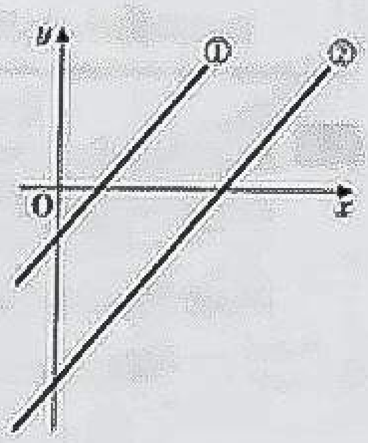
The slope of line ② is $\boxed{-2}$.

Since the two lines are parallel,

$$\frac{a}{4}=-2$$

$$\therefore\ a=-8$$

3. Find the constant a for which two lines $(a+3)x-4y=5$ …① and $(2a-1)x-2y=10$ …② are parallel.



[Sol] The slope of line ① is $\frac{a+3}{4}$.

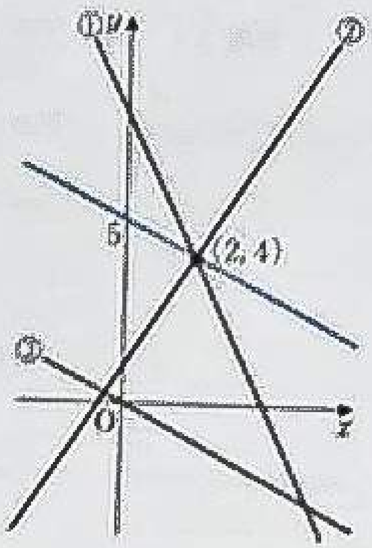
The slope of line ② is $\frac{2a-1}{2}$.

Since the two lines are parallel,

$$\frac{a+3}{4}=\frac{2a-1}{2}$$

$$\therefore\ a=\frac{5}{3}$$

4. Find the equation of the line which passes through the point of intersection of two lines $2x+y-8=0$ …① and $3x-2y+2=0$ …②, and is parallel to line $x+2y=0$ …③.



[Sol] From ① and ②, $x=2$, $y=4$

Therefore, the point of intersection is $(2, 4)$.

Also, the slope of line ③ is $-\frac{1}{2}$.

Therefore, the equation of the line is

$$y-4=-\frac{1}{2}(x-2)$$

$$\therefore\ x+2y-10=0 \quad \left[\,y=-\frac{1}{2}x+5\,\right]$$

Points and Lines 2

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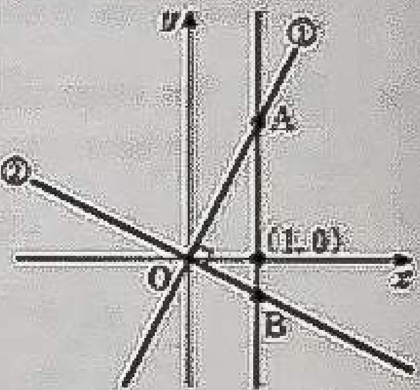
Prove that if line $y=m_1x$ ···① is perpendicular to line $y=m_2x$ ···②, then $m_1m_2=-1$.

[Sol] As shown in the diagram, draw a line parallel to the y-axis and passing through point $(1, 0)$. The points A and B, at which this line intersects lines ① and ②, are



A($\boxed{1}$, $\boxed{m_1}$)

B($\boxed{1}$, $\boxed{m_2}$)

Since $\triangle AOB$ is a right-angled triangle with $\angle AOB=90°$,

$OA^2+OB^2=AB^2$ ···③ ⟵ The Pythagorean Theorem

Therefore,

$(\boxed{1^2+m_1^2})+(\boxed{1^2+m_2^2})=\boxed{(m_1-m_2)^2}$ ···④

From ④,

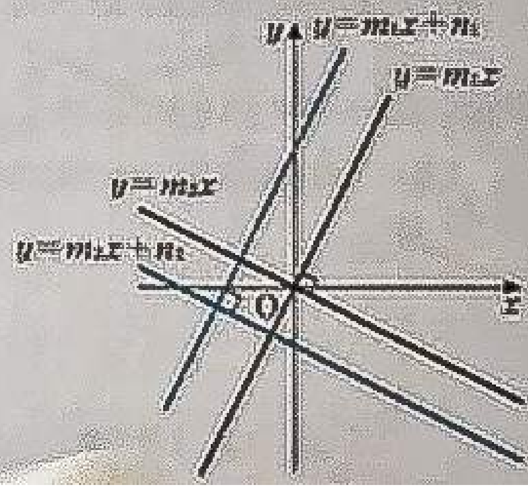
$m_1m_2=\boxed{-1}$ ···⑤ ⟵ From ④, $2+m_1^2+m_2^2=m_1^2-2m_1m_2+m_2^2$

Answers: 1, m_1, 1, m_2, $1^2+m_1^2$, $1^2+m_2^2$, $(m_1-m_2)^2$, -1

Conversely, when ⑤ is true, ④ and ③ are also true.

Therefore, two lines ① and ② are perpendicular.

Given two lines $y=m_1x+n_1$ and $y=m_2x+n_2$ parallel to lines ① and ②, $m_1m_2=-1$ is also true.



M15b

Perpendicular Condition

For two lines $y=m_1x+n_1$ and $y=m_2x+n_2$,
the perpendicular condition is $m_1m_2=-1$.

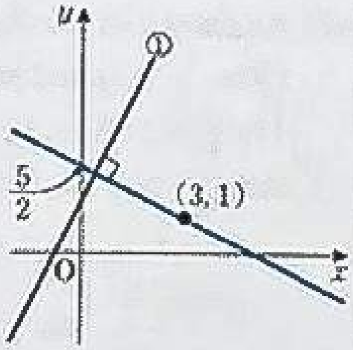
Ex Find the equation of the line which passes through point $(3, 1)$ and is perpendicular to line $4x-2y+3=0$ ···①.

[Sol] The slope of line ① is 2.
Let the slope of the line be m.

$2m=-1$, i.e. $m=-\dfrac{1}{2}$

$\therefore\ y-1=-\dfrac{1}{2}(x-3)$

$\therefore\ x+2y-5=0$



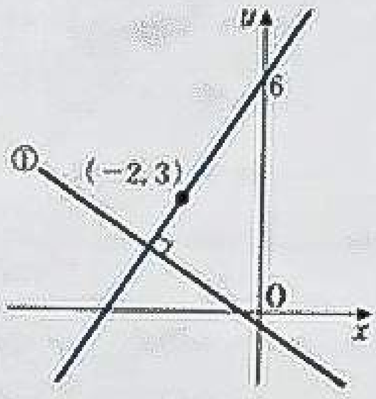
1. Find the equation of the line which passes through point $(-2, 3)$ and is perpendicular to line $2x+3y+1=0$ ···①.

[Sol] The slope of line ① is $-\dfrac{2}{3}$.
Let the slope of the line be m.

$-\dfrac{2}{3}m=-1$, i.e. $m=\dfrac{3}{2}$

$\therefore\ y-3=\dfrac{3}{2}(x+2)$

$\therefore\ 3x-2y+12=0$ $\left[y=\dfrac{3}{2}x+6\right]$



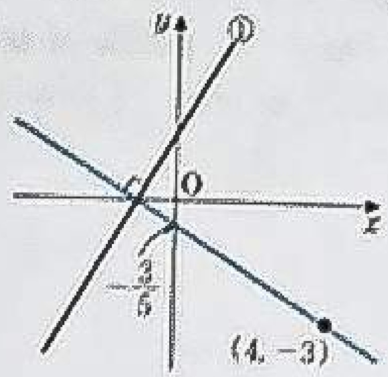
2. Find the equation of the line which passes through point $(4, -3)$ and is perpendicular to line $5x-3y+5=0$ ···①.

[Sol] The slope of line ① is $\dfrac{5}{3}$.
Let the slope of the line be m.

$\dfrac{5}{3}m=-1$, i.e. $m=-\dfrac{3}{5}$

$\therefore\ y+3=-\dfrac{3}{5}(x-4)$

$\therefore\ 3x+5y+3=0$ $\left[y=-\dfrac{3}{5}x-\dfrac{3}{5}\right]$



Points and Lines 2

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1. Find the equation of the line which passes through point $(1, 1)$ and is perpendicular to line AB which passes through points $A(1, -2)$ and $B(5, 6)$.

[Sol] The slope of line AB is

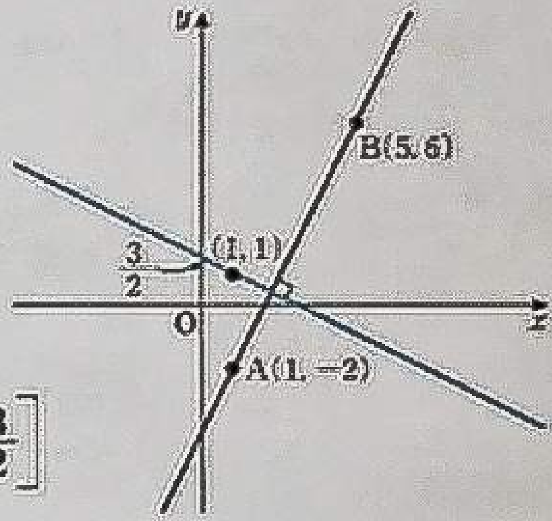
$$\frac{6+2}{5-1}=2$$

Let the slope of the line be m.

$2m=-1$, i.e. $m=-\frac{1}{2}$

$$\therefore\ y-1=-\frac{1}{2}(x-1)$$

$$\therefore\ x+2y-3=0 \quad \left[y=-\frac{1}{2}x+\frac{3}{2}\right]$$



2. Find the equation of the line which passes through the point of intersection of lines $x-3y+3=0$ ···① and $4x+2y-9=0$ ···②, and is perpendicular to line $x+2y=0$ ···③.

[Sol] From ① and ②, $x=\frac{3}{2}$, $y=\frac{3}{2}$

Therefore, the point of intersection is $\left(\frac{3}{2}, \frac{3}{2}\right)$.

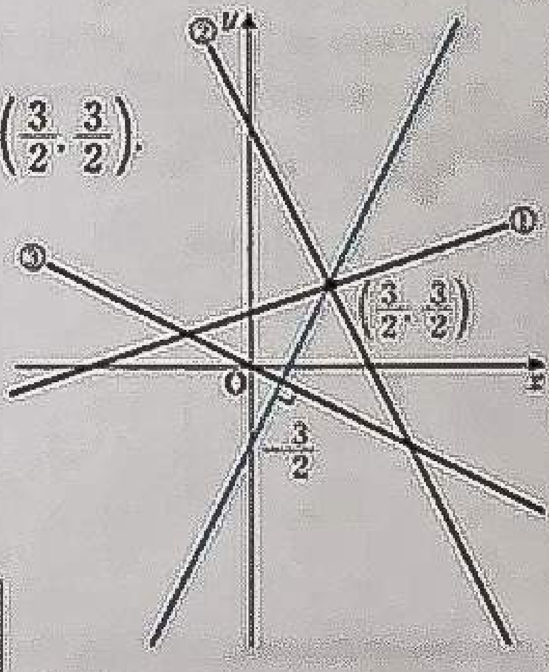
Also, the slope of line ③ is $-\frac{1}{2}$.

Let the slope of the line be m.

$-\frac{1}{2}m=-1$, i.e. $m=2$

$$\therefore\ y-\frac{3}{2}=2\left(x-\frac{3}{2}\right)$$

$$\therefore\ 4x-2y-3=0 \quad \left[y=2x-\frac{3}{2}\right]$$



M16b

3. Find a for which two lines $ax+3y+a=0$ …① and $(a-2)x-y-1=0$ …② are perpendicular.

[Sol] The slope of line ① is $\boxed{-\dfrac{1}{3}a}$.

The slope of line ② is $\boxed{a-2}$.

Since the two lines are perpendicular,

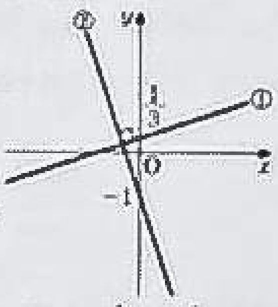
$$\left(-\frac{1}{3}a\right)\cdot(a-2)=-1$$

$$a^2-2a-3=0$$

$$(a+1)(a-3)=0$$

$$\therefore\ a=-1,\ 3$$

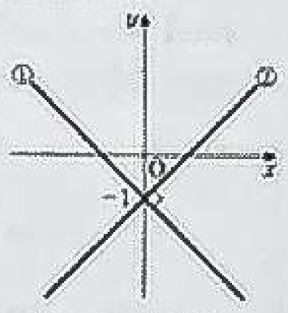
(When $a=-1$)



①: $y=\dfrac{1}{3}x+\dfrac{1}{3}$

②: $y=-3x-1$

(When $a=3$)



①: $y=-x-1$

②: $y=x-1$

. Find a for which two lines $ax-4y-2=0$ …① and $(a+3)x+y-1=0$ …② are perpendicular.

Sol] The slope of line ① is $\dfrac{1}{4}a$.

The slope of line ② is $-(a+3)$.

Since the two lines are perpendicular,

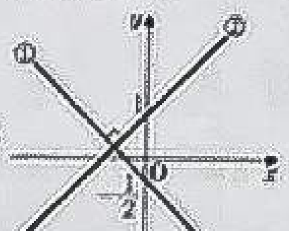
$$\frac{1}{4}a\cdot[-(a+3)]=-1$$

$$a^2+3a-4=0$$

$$(a+4)(a-1)=0$$

$$\therefore\ a=-4,\ 1$$

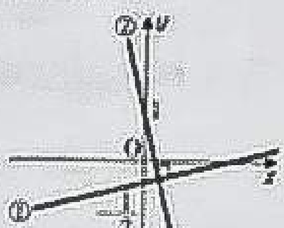
(When $a=-4$)



①: $y=-x-\dfrac{1}{2}$

②: $y=x+1$

(When $a=1$)



①: $y=\dfrac{1}{4}x-\dfrac{1}{2}$

②: $y=-4x+1$

M17a

KUMON

Points and Lines 2

Name

Date / /

Time : to :

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Ex. Given points A(0, 6) and B(4, 4), find the equation of the perpendicular bisector of line segment AB.

[Sol] The midpoint of line segment AB is

$\left(\frac{0+4}{2}, \frac{6+4}{2}\right)$, i.e. (2, 5).

Since the slope of line AB is

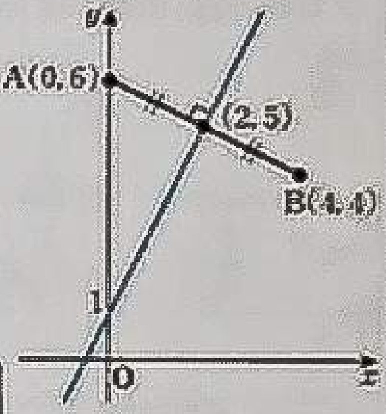
$\frac{4-6}{4-0}=-\frac{1}{2}$,

$\left(-\frac{1}{2}\right)\cdot m=-1$

the slope of the line is 2.

$\therefore\ y-5=2(x-2)$ ← Passing through midpoint (2, 5)

$\therefore\ 2x-y+1=0$



1. Given points A(4, 0) and B(−2, −2), find the equation of the perpendicular bisector of line segment AB.

[Sol] The midpoint of line segment AB is

$\left(\frac{4-2}{2}, \frac{0-2}{2}\right)$, i.e. (1, −1).

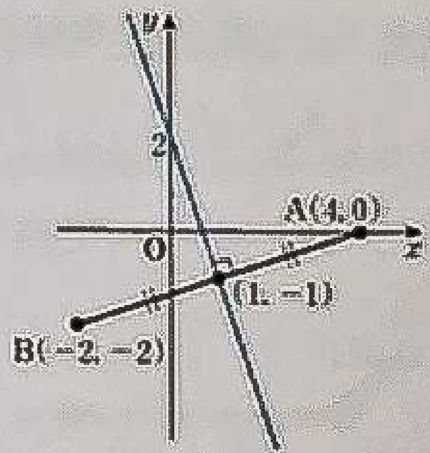
Since the slope of line AB is

$\frac{-2-0}{-2-4}=\frac{1}{3}$,

the slope of the line is −3.

$\therefore\ y+1=-3(x-1)$

$\therefore\ 3x+y-2=0$ $\quad[y=-3x+2]$



A line which passes through the midpoint of a line segment and is perpendicular to the line segment is called a ***perpendicular bisector***. A point which lies on a perpendicular bisector is equidistant from the end points of the line segment.

2. Given points A(1, 3) and B(4, −1), find the equation of the perpendicular bisector of line segment AB.

[Sol] The midpoint of line segment AB is

$\left(\frac{1+4}{2}, \frac{3-1}{2}\right)$, i.e. $\left(\frac{5}{2}, 1\right)$.

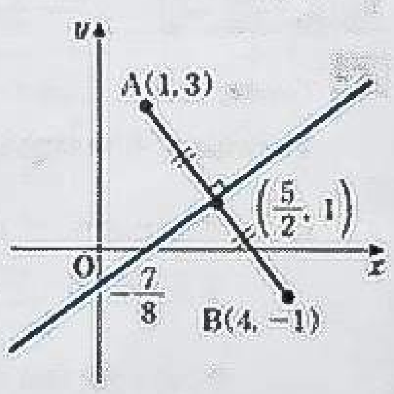
Since the slope of line AB is

$\frac{-1-3}{4-1}=-\frac{4}{3}$,

the slope of the line is $\frac{3}{4}$.

$\therefore\ y-1=\frac{3}{4}\left(x-\frac{5}{2}\right)$

$\therefore\ 6x-8y-7=0\quad \left[y=\frac{3}{4}x-\frac{7}{8}\right]$



3. Given points A(1, 3) and B(−2, 2), find the equation of the perpendicular bisector of line segment AB.

[Sol] The midpoint of line segment AB is

$\left(\frac{1-2}{2}, \frac{3+2}{2}\right)$, i.e. $\left(-\frac{1}{2}, \frac{5}{2}\right)$.

Since the slope of line AB is

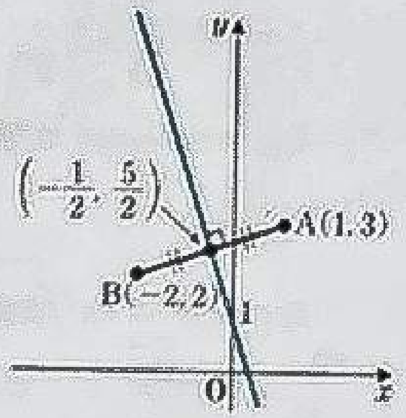
$\frac{2-3}{-2-1}=\frac{1}{3}$,

the slope of the line is −3.

$\therefore\ y-\frac{5}{2}=-3\left(x+\frac{1}{2}\right)$

$\therefore\ 3x+y-1=0\quad [y=-3x+1]$

Generally, coefficients have to be simpler integers; therefore, divide both sides by 2 when $6x+2y-2=0$.



Points and Lines 2

Name

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Ex. Find the coordinates of point $B(a, b)$ which is symmetric to point $A(-1, 4)$ with respect to line $2x-y+1=0$ ···①.

[Sol] Since the slope of line ① is 2 and line AB is perpendicular to line ①,

$$2\cdot\frac{b-4}{a+1}=-1$$

So, $a+2b-7=0$ ···②

Also, since the midpoint of line segment AB $\left(\frac{-1+a}{2}, \frac{4+b}{2}\right)$ lies on line ①,

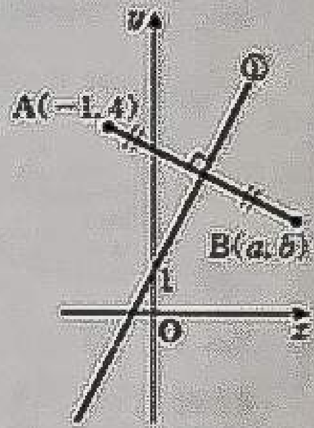
$$2\cdot\frac{-1+a}{2}-\frac{4+b}{2}+1=0$$

Substituting $x=\frac{-1+a}{2}$ and $y=\frac{4+b}{2}$ into ①

So, $2a-b-4=0$ ···③

From ② and ③, $a=3$, $b=2$

$\therefore$ B(3, 2)



1. Find the coordinates of point $B(a, b)$ which is symmetric to point $A(3, 6)$ with respect to line $2x+y-2=0$ ···①.

[Sol] Since the slope of line ① is -2 and line AB is perpendicular to line ①,

$$-2\cdot\frac{b-6}{a-3}=-1$$

So, $a-2b+9=0$ ···②

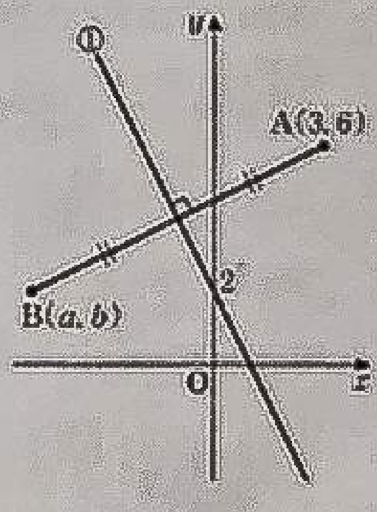
Also, since the midpoint of line segment AB $\left(\frac{3+a}{2}, \frac{6+b}{2}\right)$ lies on line ①,

$$2\cdot\frac{3+a}{2}+\frac{6+b}{2}-2=0$$

So, $2a+b+8=0$ ···③

From ② and ③, $a=-5$, $b=2$

$\therefore$ B(−5, 2)



M18b

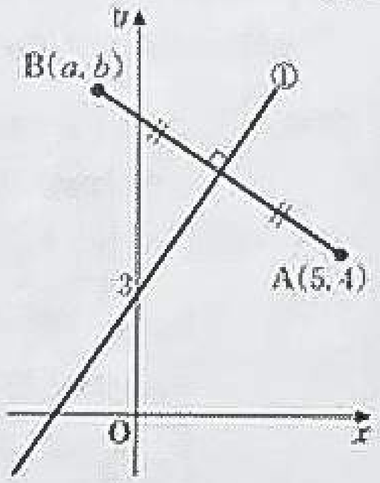
2. Find the coordinates of point $B(a, b)$ which is symmetric to point $A(5, 4)$ with respect to line $3x-2y+6=0$ …①.

[Sol] Since the slope of line ① is $\frac{3}{2}$ and line AB is perpendicular to line ①,

$$\frac{3}{2}\cdot\frac{b-4}{a-5}=-1$$

So, $2a+3b-22=0$ …②

Also, since the midpoint of line segment AB $\left(\frac{5+a}{2}, \frac{4+b}{2}\right)$ lies on line ①,



$$3\cdot\frac{5+a}{2}-2\cdot\frac{4+b}{2}+6=0$$

So, $3a-2b+19=0$ …③

From ② and ③, $a=-1$, $b=8$

$\therefore B(-1, 8)$

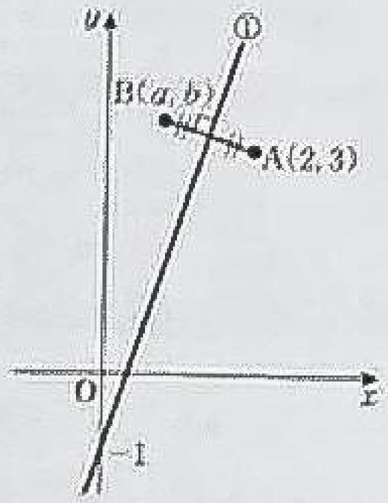
3. Find the coordinates of point $B(a, b)$ which is symmetric to point $A(2, 3)$ with respect to line $3x-y-1=0$ …①.

[Sol] Since the slope of line ① is 3 and line AB is perpendicular to line ①,

$$3\cdot\frac{b-3}{a-2}=-1$$

So, $a+3b-11=0$ …②

Also, since the midpoint of line segment AB $\left(\frac{2+a}{2}, \frac{3+b}{2}\right)$ lies on line ①,



$$3\cdot\frac{2+a}{2}-\frac{3+b}{2}-1=0$$

So, $3a-b+1=0$ …③

From ② and ③, $a=\frac{4}{5}$, $b=\frac{17}{5}$

$\therefore B\left(\frac{4}{5}, \frac{17}{5}\right)$

Points and Lines 2

Name

Date / /

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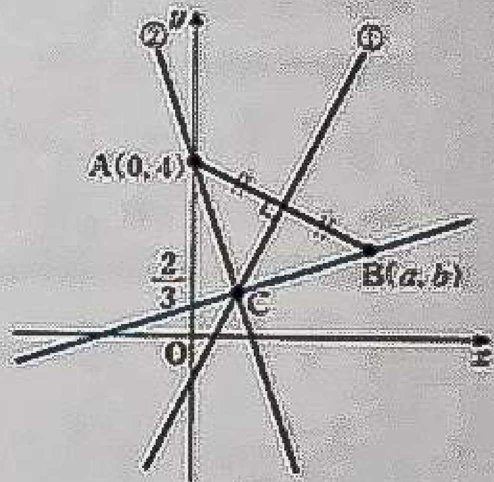
Time : to :

1. As shown in the diagram, given two lines $2x-y-1=0$ ···① and $3x+y-4=0$ ···②, and point A(0, 4) lying on line ②, find the equation of the line which is symmetric to line ② with respect to line ①.



[Sol] Let point B which is symmetric to point A with respect to line ① be (a, b).

Since the slope of line ① is 2 and line AB is perpendicular to line ①,

$$2\cdot\frac{b-4}{a-0}=-1$$

So, $a+2b-8=0$ ···③

Since the midpoint of line segment AB $\left(\frac{a}{2}, \frac{4+b}{2}\right)$ lies on line ①,

$$2\cdot\frac{a}{2}-\frac{4+b}{2}-1=0$$

So, $2a-b-6=0$ ···④

From ③ and ④, $a=4, b=2$

$\therefore$ B(4, 2)

Also, let the intersection of lines ① and ② be point C.

From ① and ②, $x=1, y=1$

$\therefore$ C(1, 1)

Since the line passes through two points B and C,

$$y-2=\frac{1-2}{1-4}(x-4)$$

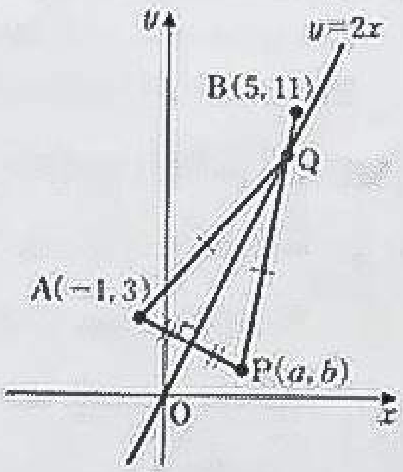
$$\therefore x-3y+2=0 \quad \left[y=\frac{1}{3}x+\frac{2}{3}\right]$$

M19b

2. Given points A$(-1, 3)$ and B$(5, 11)$, solve the following questions.

(1) Find the coordinates of point P which is symmetric to point A with respect to line $y=2x$.

[Sol] Let point P which is symmetric to point A with respect to $y=2x$ be (a, b).



Since the slope of $y=2x$ is 2 and line AP is perpendicular to $y=2x$,

$$2\cdot\frac{b-3}{a+1}=-1$$

So, $a+2b-5=0$ …①

Since the midpoint of line segment AP $\left(\frac{-1+a}{2}, \frac{3+b}{2}\right)$ lies on $y=2x$,

$$\frac{3+b}{2}=2\cdot\frac{-1+a}{2}$$

So, $2a-b-5=0$ …②

From ① and ②, $a=3, b=1$

$\therefore$ P$(3, 1)$

(2) Given that point Q lies on line $y=2x$, find the coordinates of point Q which minimizes the value of QA+QB.

[Sol] Points A and B lie on the same side with respect to $y=2x$.

Since $QA+QB=QP+QB\geqq PB$, ← Point Q lying on the perpendicular bisector of line segment AP is equidistant from points A and P. $\therefore$ QA=QP

QA+QB is minimized when three points P, Q and B lie on the same line.

Also, the equation of line PB is

$$y-1=\frac{11-1}{5-3}(x-3)$$ ← Passing through points P(3, 1) and B(5, 11)

So, $y=5x-14$ …③

From ③ and $y=2x$, $x=\frac{14}{3}, y=\frac{28}{3}$

$\therefore$ $Q\left(\frac{14}{3}, \frac{28}{3}\right)$

Points and Lines 2

Name

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Time : to :

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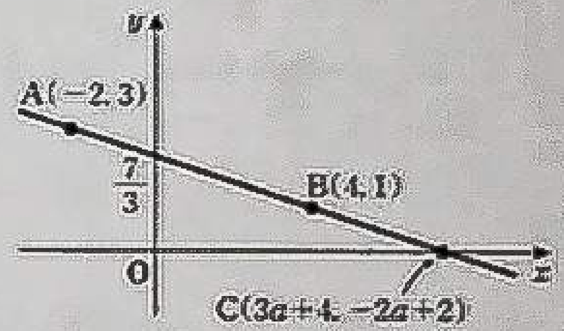
1. Given points A$(-2, 3)$, B$(4, 1)$ and C$(3a+4, -2a+2)$, find a for which all three given points lie on the same line. ➡ M12

[Sol] The equation of line AB is



$$y-3=\frac{1-3}{4+2}(x+2)$$

So, $y=-\frac{1}{3}x+\frac{7}{3}$ ⋯①

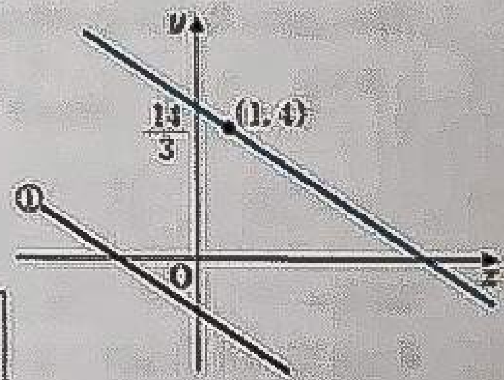
Since line ① passes through point C,

$$-2a+2=-\frac{1}{3}(3a+4)+\frac{7}{3}$$

$\therefore\ a=1$

2. Find the equation of the line which passes through point $(1, 4)$ and is parallel to line $2x+3y+5=0$ ⋯①. ➡ M14

[Sol] The slope of line ① is $-\frac{2}{3}$.



Therefore, the equation of the line is

$$y-4=-\frac{2}{3}(x-1)$$

$\therefore\ 2x+3y-14=0\quad\left[y=-\frac{2}{3}x+\frac{14}{3}\right]$

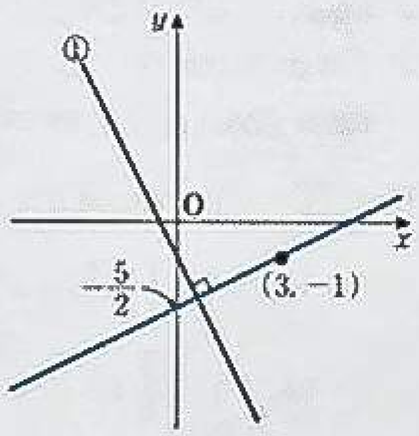
3. Find the equation of the line which passes through point $(3, -1)$ and is perpendicular to line $2x+y+1=0$ ⋯①. ➡ M15

[Sol] Since the slope of line ① is -2,

the slope of the line is $\frac{1}{2}$.

$\therefore\ y+1=\frac{1}{2}(x-3)$

$\therefore\ x-2y-5=0 \quad \left[y=\frac{1}{2}x-\frac{5}{2}\right]$



4. Find the coordinates of point $B(a, b)$ which is symmetric to point $A(4, -1)$ with respect to line $4x-2y-3=0$ ⋯①. ➡ M18

[Sol] Since the slope of line ① is 2 and line AB is perpendicular to line ①,

$2\cdot\frac{b+1}{a-4}=-1$

So, $a+2b-2=0$ ⋯②

Also, since the midpoint of line segment AB

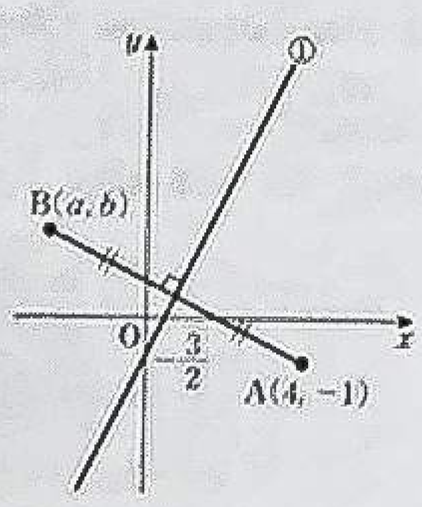
$\left(\frac{4+a}{2}, \frac{-1+b}{2}\right)$ lies on line ①,

$4\cdot\frac{4+a}{2}-2\cdot\frac{-1+b}{2}-3=0$

So, $2a-b+6=0$ ⋯③

From ② and ③, $a=-2, b=2$

$\therefore\ B(-2, 2)$



Points and Lines 3

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Time : to :

Given $\triangle ABC$ with vertices $A(-1, 7)$, $B(-3, -7)$ and $C(8, 4)$, let AP, BQ and CR be the perpendiculars dropped from A, B and C to their opposite sides of $\triangle ABC$.

(1) Find the equation of perpendicular AP.

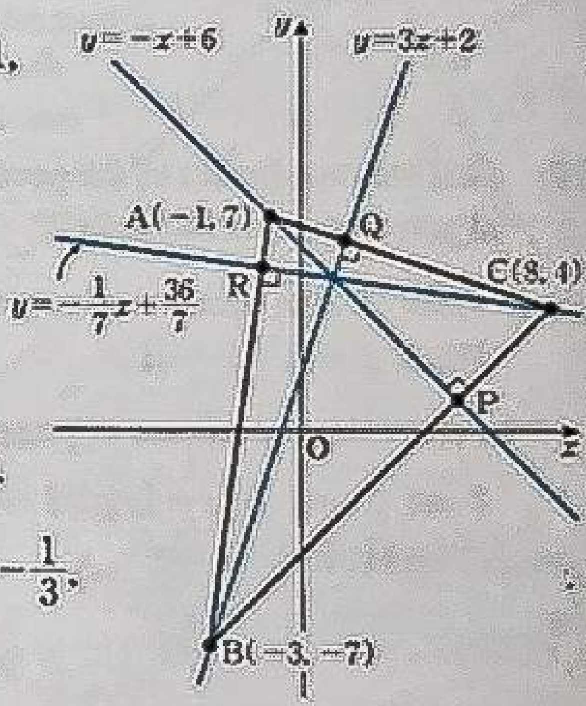
[Sol] Since the slope of line BC is $\frac{4+7}{8+3}=1$,

the slope of perpendicular AP is -1.

$\therefore\ y-7=-1\cdot(x+1)$

$\therefore\ y=-x+6$

$[x+y-6=0]$



(2) Find the equation of perpendicular BQ.

[Sol] Since the slope of line AC is $\frac{4-7}{8+1}=-\frac{1}{3}$,

the slope of perpendicular BQ is 3.

$\therefore\ y+7=3(x+3)$

$\therefore\ y=3x+2$ $[3x-y+2=0]$

(3) Find the equation of perpendicular CR.

[Sol] Since the slope of line AB is $\frac{-7-7}{-3+1}=7$,

the slope of perpendicular CR is $-\frac{1}{7}$.

$\therefore\ y-4=-\frac{1}{7}(x-8)$

$\therefore\ y=-\frac{1}{7}x+\frac{36}{7}$ $[x+7y-36=0]$

M2Ib

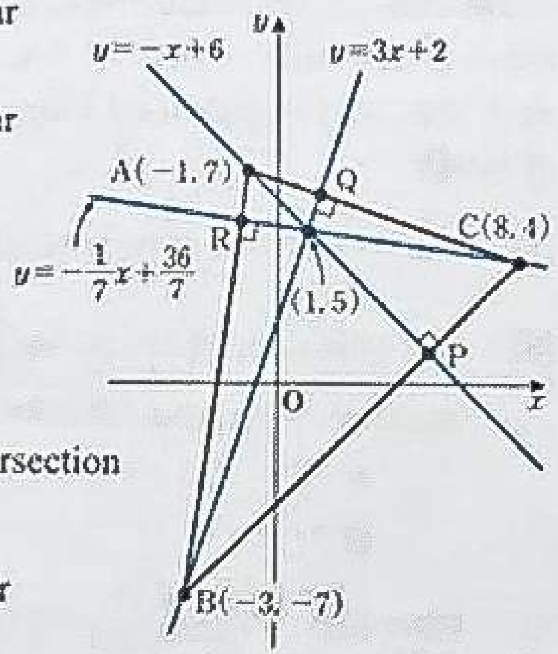
(4) Find the coordinates of the point of intersection of perpendiculars AP and BQ.

[Sol] From (1), the equation of perpendicular AP is $y=-x+6$ …①.

From (2), the equation of perpendicular BQ is $y=3x+2$ …②.

From ① and ②, $x=1$, $y=5$

$\therefore$ (1, 5)



(5) Find the coordinates of the point of intersection of perpendiculars BQ and CR.

[Sol] From (2), the equation of perpendicular BQ is $y=3x+2$ …①.

From (3), the equation of perpendicular CR is $y=-\frac{1}{7}x+\frac{36}{7}$ …②.

From ① and ②, $x=1$, $y=5$

$\therefore$ (1, 5)

(6) Find the coordinates of the point of intersection of perpendiculars CR and AP.

[Sol] From (3), the equation of perpendicular CR is $y=-\frac{1}{7}x+\frac{36}{7}$ …①.

From (1), the equation of perpendicular AP is $y=-x+6$ …②.

From ① and ②, $x=1$, $y=5$

$\therefore$ (1, 5)

From (4)–(6), perpendiculars AP, BQ and CR dropped from the vertices of △ABC to their opposite sides intersect at one point.

Points and Lines 3

Name

Date / /

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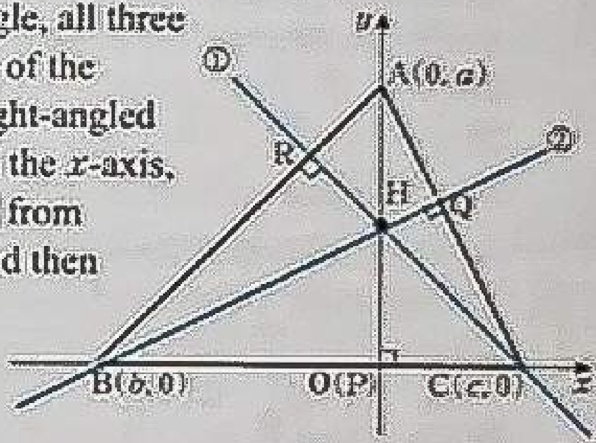
Time : to :

1. Prove that perpendiculars AP, BQ and CR dropped from the vertices of $\triangle ABC$ to their opposite sides intersect at one point.



[Sol] When $\triangle ABC$ is a right-angled triangle, all three perpendiculars intersect at the vertex of the right angle. When $\triangle ABC$ is not a right-angled triangle, place side BC of $\triangle ABC$ on the x-axis, and place perpendicular AP dropped from vertex A to side BC on the y-axis, and then let A, B and C be $A(0, a)$, $B(b, 0)$ and $C(c, 0)$ respectively. ($a \neq 0, b \neq 0, c \neq 0$)

Since the slope of line AB is $-\frac{a}{b}$,

the slope of perpendicular CR is $\boxed{\frac{b}{a}}$.

Therefore, the equation of perpendicular CR is

$y = \boxed{\frac{b}{a}x - \frac{bc}{a}}$ ...① ← Passing through point $C(c, 0)$, $y - 0 = \frac{b}{a}(x - c)$

Similarly, since the slope of line AC is

$-\frac{a}{c}$, the slope of perpendicular BQ is $\boxed{\frac{c}{a}}$.

Therefore, the equation of perpendicular BQ is

$y = \boxed{\frac{c}{a}x - \frac{bc}{a}}$ ...② ← Passing through point $B(b, 0)$, $y - 0 = \frac{c}{a}(x - b)$

Let H be the point of intersection of lines ① and ②.

$H\left(\boxed{0}, \boxed{-\frac{bc}{a}}\right)$ ← From ① and ②,
$$\frac{b}{a}x - \frac{bc}{a} = \frac{c}{a}x - \frac{bc}{a}$$
$$\frac{b-c}{a}x = 0$$
$$\therefore x = 0$$
Substituting $x = 0$ into ①,
$$y = -\frac{bc}{a}$$

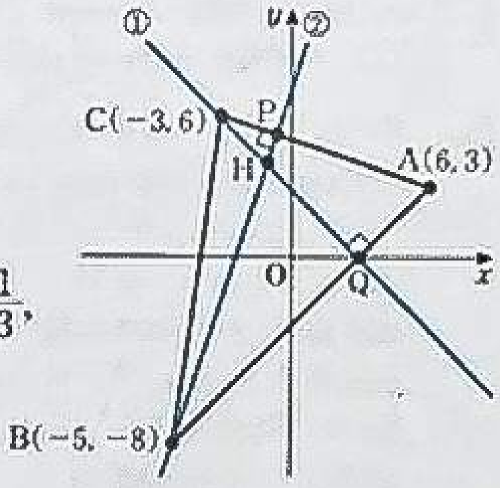
This shows that H is on the y-axis, i.e. on perpendicular AP.

Thus, three perpendiculars AP, BQ and CR intersect at one point.

The point at which all three perpendiculars of a triangle dropped from the vertices intersect is called the *orthocenter* of the triangle.

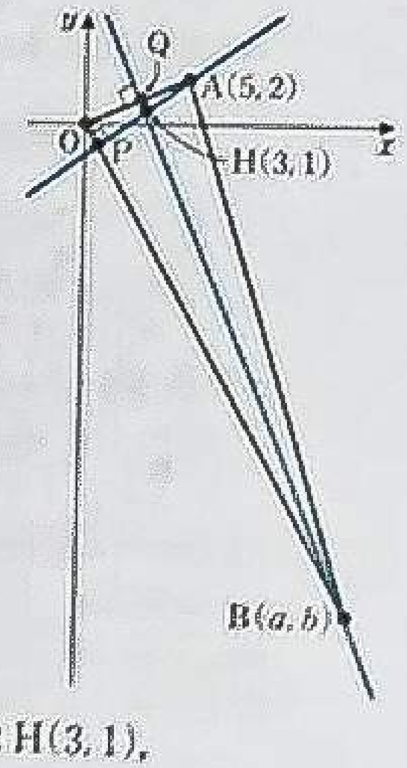
M22b

2. Given $\triangle ABC$ with vertices $A(6, 3)$, $B(-5, -8)$ and $C(-3, 6)$, let BP and CQ be the perpendiculars dropped from B and C to their opposite sides of $\triangle ABC$. Find the coordinates of orthocenter H of $\triangle ABC$.



[Sol] Since the slope of line AB is $\frac{-8-3}{-5-6}=1$,
the slope of perpendicular CQ is -1.
Therefore, the equation of perpendicular CQ is $y-6=-1\cdot(x+3)$.
So, $y=-x+3$ ⋯①
Since the slope of line AC is $\frac{6-3}{-3-6}=-\frac{1}{3}$,
the slope of perpendicular BP is 3.
Therefore, the equation of perpendicular BP is $y+8=3(x+5)$.
So, $y=3x+7$ ⋯②
From ① and ②, $x=-1$, $y=4$
$\therefore$ $H(-1, 4)$

3. Given $\triangle OAB$ with vertices $O(0, 0)$, $A(5, 2)$ and $B(a, b)$, let AP and BQ be the perpendiculars dropped from A and B to their opposite sides of $\triangle OAB$ $(a \neq 0, b \neq 0)$. Find the constants a and b when the orthocenter of $\triangle OAB$ is $H(3, 1)$.



[Sol] Since the slope of line OA is $\frac{2-0}{5-0}=\frac{2}{5}$,
the slope of perpendicular BQ is $-\frac{5}{2}$.
Therefore, the equation of perpendicular BQ is $y-b=-\frac{5}{2}(x-a)$.
Also, since perpendicular BQ passes through point $H(3, 1)$,
$5a+2b=17$ ⋯①
Since the slope of line OB is $\frac{b-0}{a-0}=\frac{b}{a}$,
the slope of perpendicular AP is $-\frac{a}{b}$.
Therefore, the equation of perpendicular AP is $y-2=-\frac{a}{b}(x-5)$.
Also, since perpendicular AP passes through point $H(3, 1)$,
$2a+b=0$ ⋯②
From ① and ②, $a=17$, $b=-34$

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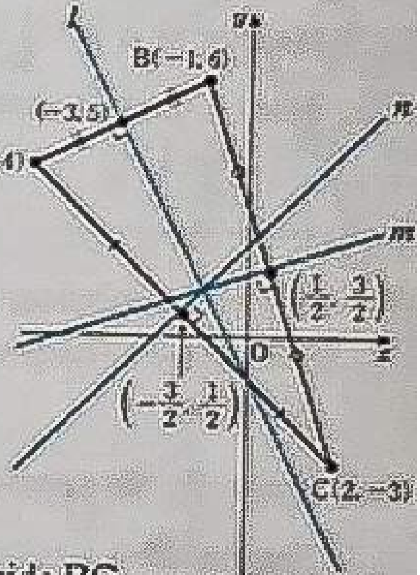
Given $\triangle ABC$ with vertices $A(-5, 4)$, $B(-1, 6)$ and $C(2, -3)$, let l, m and n be the perpendicular bisectors of sides AB, BC and CA respectively.



(1) Find the equation of perpendicular bisector l of side AB.

[Sol] The midpoint of side AB is

$\left(\frac{-5-1}{2}, \frac{4+6}{2}\right)$, i.e. $(-3, 5)$.

Since the slope of line AB is $\frac{6-4}{-1+5}=\frac{1}{2}$, the slope of perpendicular bisector l is -2.

Therefore, the equation of perpendicular bisector l is $y-5=-2(x+3)$.

$\therefore\ y=-2x-1 \quad [2x+y+1=0]$

(2) Find the equation of perpendicular bisector m of side BC.

[Sol] The midpoint of side BC is

$\left(\frac{-1+2}{2}, \frac{6-3}{2}\right)$, i.e. $\left(\frac{1}{2}, \frac{3}{2}\right)$.

Since the slope of line BC is $\frac{-3-6}{2+1}=-3$,

the slope of perpendicular bisector m is $\frac{1}{3}$.

Therefore, the equation of perpendicular bisector m is $y-\frac{3}{2}=\frac{1}{3}\left(x-\frac{1}{2}\right)$.

$\therefore\ y=\frac{1}{3}x+\frac{4}{3} \quad [x-3y+4=0]$

(3) Find the equation of perpendicular bisector n of side CA.

[Sol] The midpoint of side CA is

$\left(\frac{2-5}{2}, \frac{-3+4}{2}\right)$, i.e. $\left(-\frac{3}{2}, \frac{1}{2}\right)$.

Since the slope of line CA is $\frac{4+3}{-5-2}=-1$,

the slope of perpendicular bisector n is 1.

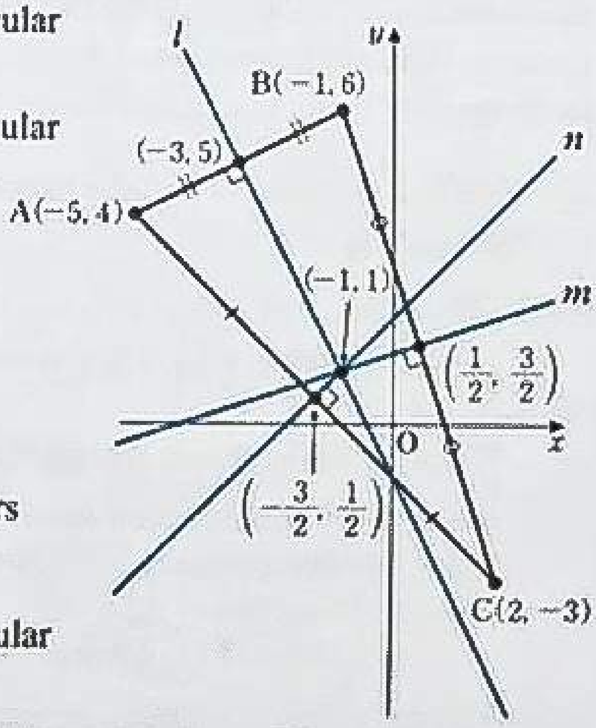
Therefore, the equation of perpendicular bisector n is $y-\frac{1}{2}=1\cdot\left(x+\frac{3}{2}\right)$.

$\therefore\ y=x+2 \quad [x-y+2=0]$

M23b

(4) Find the coordinates of the point of intersection of perpendicular bisectors l and m.

[Sol] From (1), the equation of perpendicular bisector l is $y=-2x-1$ …①.
From (2), the equation of perpendicular bisector m is $y=\frac{1}{3}x+\frac{4}{3}$ …②.
From ① and ②, $x=-1$, $y=1$
$\therefore$ $(-1, 1)$



(5) Find the coordinates of the point of intersection of perpendicular bisectors m and n.

[Sol] From (2), the equation of perpendicular bisector m is $y=\frac{1}{3}x+\frac{4}{3}$ …①.
From (3), the equation of perpendicular bisector n is $y=x+2$ …②.
From ① and ②, $x=-1$, $y=1$
$\therefore$ $(-1, 1)$

(6) Find the coordinates of the point of intersection of perpendicular bisectors n and l.

[Sol] From (3), the equation of perpendicular bisector n is $y=x+2$ …①.
From (1), the equation of perpendicular bisector l is $y=-2x-1$ …②.
From ① and ②, $x=-1$, $y=1$
$\therefore$ $(-1, 1)$

From (4)–(6), perpendicular bisectors l, m and n of sides AB, BC and CA of $\triangle$ABC intersect at one point.

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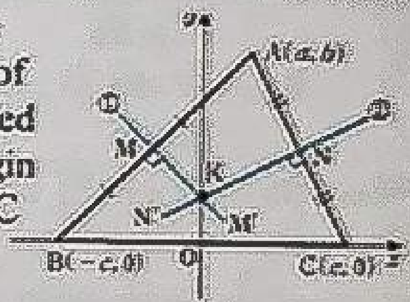
1. Prove that all the perpendicular bisectors of three sides of $\triangle ABC$ intersect at one point.

[Sol] When $\triangle ABC$ is a right-angled triangle, all three perpendicular bisectors intersect at the midpoint of the hypotenuse. When $\triangle ABC$ is not a right-angled triangle, place the midpoint of side BC at the origin and side BC on the x-axis, and then let A, B and C be $A(a, b)$, $B(-c, 0)$ and $C(c, 0)$. $(a \neq \pm c, b \neq 0, c \neq 0)$



Midpoint M of side AB is $\left(\boxed{\dfrac{a-c}{2}}, \boxed{\dfrac{b}{2}}\right)$.

Since the slope of line AB is $\boxed{\dfrac{b}{a+c}}$,

the slope of perpendicular bisector MM′ of side AB is $\boxed{-\dfrac{a+c}{b}}$.

Therefore, the equation of perpendicular bisector MM′ is

$$y-\boxed{\frac{b}{2}}=\boxed{-\frac{a+c}{b}}\left(x-\boxed{\frac{a-c}{2}}\right) \cdots ①$$

Substituting $x=0$ into ①, $y-\dfrac{b}{2}=-\dfrac{a+c}{b}\left(0-\dfrac{a-c}{2}\right)$, I.e. $y=\dfrac{a+c}{b}\cdot\dfrac{a-c}{2}+\dfrac{b}{2}$

Let K be the point of intersection of ① and the y-axis.

$$K\left(\boxed{0}, \boxed{\frac{a^2+b^2-c^2}{2b}}\right)$$

Similarly, the equation of perpendicular bisector NN′ of side AC is

$$y-\boxed{\frac{b}{2}}=\boxed{-\frac{a-c}{b}}\left(x-\boxed{\frac{a+c}{2}}\right) \cdots ②$$

The coordinates of midpoint N of side AC are $\left(\dfrac{a+c}{2}, \dfrac{b}{2}\right)$. Also, the slope of line AC is $\dfrac{b}{a-c}$. Therefore, the slope of perpendicular bisector NN′ is $-\dfrac{a-c}{b}$.

The coordinates of the point of intersection of ② and the y-axis are $\left(\boxed{0}, \boxed{\dfrac{a^2+b^2-c^2}{2b}}\right)$.

Since this coincides with K,
perpendicular bisectors MM′ and NN′ intersect at one point on the y-axis. Since the perpendicular bisector of side BC is the y-axis, all the perpendicular bisectors intersect at one point.

The point at which all the perpendicular bisectors of three sides of a triangle intersect is called the ***circumcenter*** of the triangle.

M24b

2. Given three lines $3x+2y-16=0$ ⋯①, $2x-y+1=0$ ⋯② and $x-4y+4=0$ ⋯③, let A be the point of intersection of lines ① and ②, B be the point of intersection of lines ② and ③, and C be the point of intersection of lines ③ and ①. Find the coordinates of circumcenter K of △ABC.

[Sol] From ① and ②, $x=2$, $y=5$

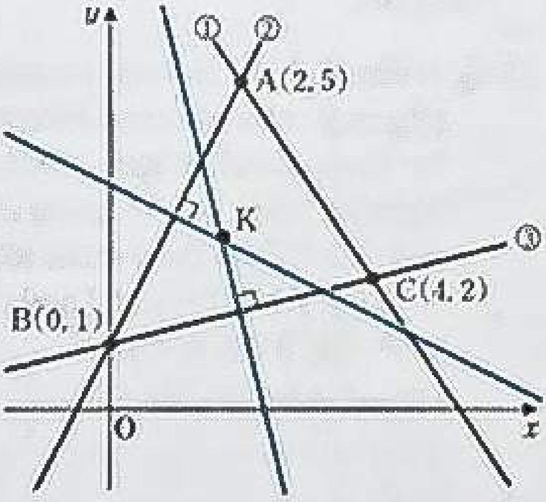
$\therefore$ A(2, 5)

From ② and ③, $x=0$, $y=1$

$\therefore$ B(0, 1)

From ③ and ①, $x=4$, $y=2$

$\therefore$ C(4, 2)



Also, the midpoint of side AB is $\left(\frac{2+0}{2}, \frac{5+1}{2}\right)$, i.e. (1, 3).

From ②, the slope of line AB is 2.

The slope of the perpendicular bisector of side AB is $-\frac{1}{2}$.

Therefore, the equation of the perpendicular bisector of side AB is

$$y-3=-\frac{1}{2}(x-1)$$

So, $y=-\frac{1}{2}x+\frac{7}{2}$ ⋯④

The midpoint of side BC is $\left(\frac{0+4}{2}, \frac{1+2}{2}\right)$, i.e. $\left(2, \frac{3}{2}\right)$.

From ③, the slope of line BC is $\frac{1}{4}$.

The slope of the perpendicular bisector of side BC is -4.

Therefore, the equation of the perpendicular bisector of side BC is

$$y-\frac{3}{2}=-4(x-2)$$

So, $y=-4x+\frac{19}{2}$ ⋯⑤

From ④ and ⑤, $x=\frac{12}{7}$, $y=\frac{37}{14}$

$\therefore$ K$\left(\frac{12}{7}, \frac{37}{14}\right)$

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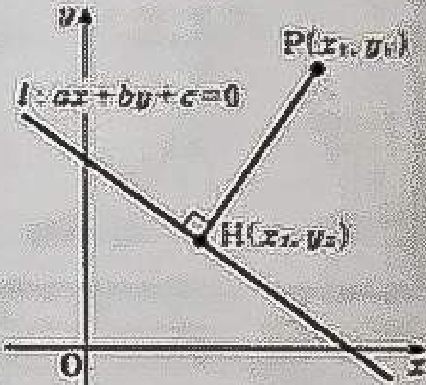
Given line $l: ax+by+c=0$ $(a\neq 0, b\neq 0)$ and point $P(x_1, y_1)$ which is not on line l, prove that the length of perpendicular PH dropped from point P to point $H(x_2, y_2)$ which is on line l is $PH=\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$.



[Sol] The slope of line $l: ax+by+c=0$ is $-\dfrac{a}{b}$.

Therefore, the slope of perpendicular PH is $\dfrac{b}{a}$.

$\therefore \dfrac{y_2-y_1}{x_2-x_1}=\dfrac{b}{a}$ …①

Rearranging ①, $\dfrac{x_2-x_1}{a}=\dfrac{y_2-y_1}{b}$

Let k be the value of this expression.

$x_2-x_1=ak,\ y_2-y_1=bk$ …② ⟵ Considering $\dfrac{x_2-x_1}{a}=\dfrac{y_2-y_1}{b}=k$

Also, since point $H(x_2, y_2)$ is on line l,

$ax_2+by_2+c=0$ …③

From ② and ③, $a(x_1+ak)+b(y_1+bk)+c=0$ ⟵ From ②, substituting $x_2=x_1+ak$ and $y_2=y_1+bk$ into ③

$(a^2+b^2)k+(ax_1+by_1+c)=0$

$\therefore k=\boxed{-\dfrac{ax_1+by_1+c}{a^2+b^2}}$ …④

From ② and ④, $PH^2=(x_2-x_1)^2+(y_2-y_1)^2=(\boxed{a^2+b^2})k^2$

$=(\boxed{a^2+b^2})\cdot\left(\boxed{-\dfrac{ax_1+by_1+c}{a^2+b^2}}\right)^2$ From ④

$=\boxed{\dfrac{(ax_1+by_1+c)^2}{a^2+b^2}}$

When $A^2=B^2$ and $A>0$, $A=|B|$

$\therefore PH=\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

Answers: $-\dfrac{ax_1+by_1+c}{a^2+b^2}$, a^2+b^2, a^2+b^2, $-\dfrac{ax_1+by_1+c}{a^2+b^2}$, $\dfrac{(ax_1+by_1+c)^2}{a^2+b^2}$

Distance from a Point to a Line

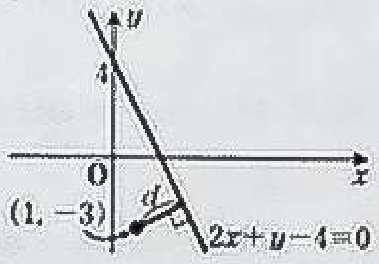
The distance d from point (x_1, y_1) to line $ax+by+c=0$ is

$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

Find the distance d from a point to a line for the following questions.

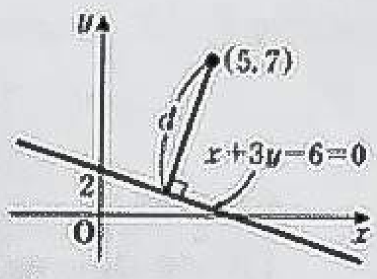
(1) Point $(1, -3)$ and line $2x+y-4=0$

[Sol] $d=\dfrac{|2\cdot1-3-4|}{\sqrt{2^2+1^2}}=\sqrt{5}$



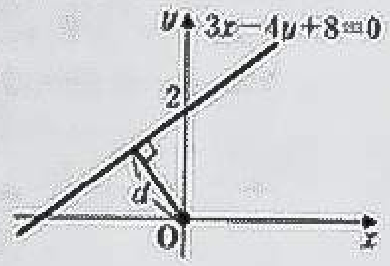
(2) Point $(5, 7)$ and line $x+3y-6=0$

[Sol] $d=\dfrac{|5+3\cdot7-6|}{\sqrt{1^2+3^2}}=\dfrac{20}{\sqrt{10}}=2\sqrt{10}$



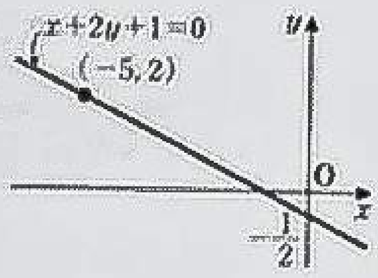
(3) Point $(0, 0)$ and line $3x-4y+8=0$

[Sol] $d=\dfrac{|3\cdot0-4\cdot0+8|}{\sqrt{3^2+(-4)^2}}=\dfrac{8}{5}$



(4) Point $(-5, 2)$ and line $x+2y+1=0$

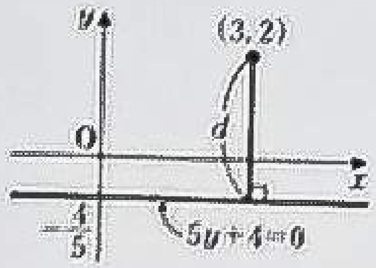
[Sol] $d=\dfrac{|-5+2\cdot2+1|}{\sqrt{1^2+2^2}}=0$



(5) Point $(3, 2)$ and line $5y+4=0$

[Sol] $d=\dfrac{|0\cdot3+5\cdot2+4|}{\sqrt{0^2+5^2}}=\dfrac{14}{5}$

[Alternative Solution $d=2+\dfrac{4}{5}=\dfrac{14}{5}$]



The formula above is also true when either a or b is 0.

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Ex. Find the coordinates of point P which is on the x-axis and at a distance of $\sqrt{5}$ units from line $2x-y+1=0$.

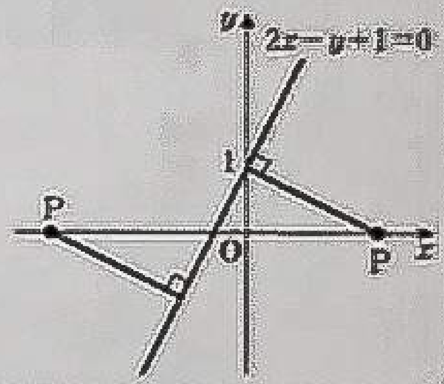
[Sol] Let P be $P(p, 0)$.

$$\sqrt{5}=\frac{|2\cdot p-0+1|}{\sqrt{2^2+(-1)^2}}$$

$\therefore\ |2p+1|=5$ $\quad \boxed{2p+1=\pm5}$

$p=2,\ -3$

$\therefore\ P(2, 0), (-3, 0)$



1. Find the coordinates of point P which is on the x-axis and at a distance of $\sqrt{10}$ units from line $3x-y+2=0$.

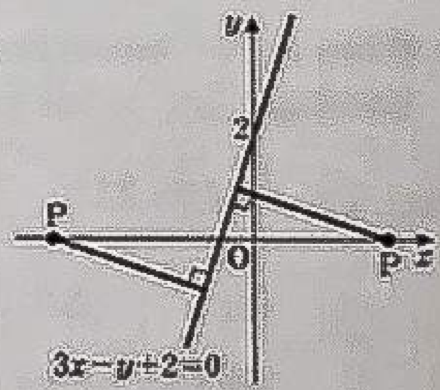
[Sol] Let P be $P(p, 0)$.

$$\sqrt{10}=\frac{|3\cdot p-0+2|}{\sqrt{3^2+(-1)^2}}$$

$\therefore\ |3p+2|=10$

$p=\frac{8}{3},\ -4$

$\therefore\ P\left(\frac{8}{3}, 0\right), (-4, 0)$



2. Find the coordinates of point P which is on the y-axis and at a distance of 2 units from line $3x+2y-6=0$.

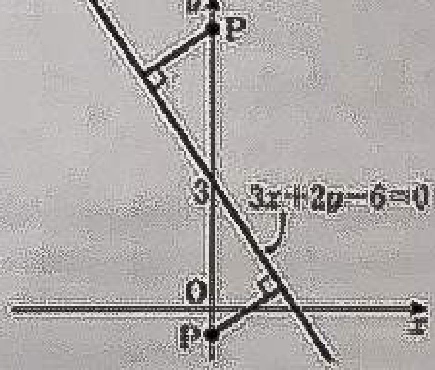
[Sol] Let P be $P(0, p)$.

$$2=\frac{|3\cdot0+2\cdot p-6|}{\sqrt{3^2+2^2}}$$

$\therefore\ |2p-6|=2\sqrt{13}$

$p=3\pm\sqrt{13}$

$\therefore\ P(0, 3\pm\sqrt{13})$



M26b

3. Find a for which the distance from point $(5, -1)$ to line $ax-4y-4=0$ is 3.

[Sol] $3=\dfrac{|a\cdot5-4\cdot(-1)-4|}{\sqrt{a^2+(-4)^2}}$

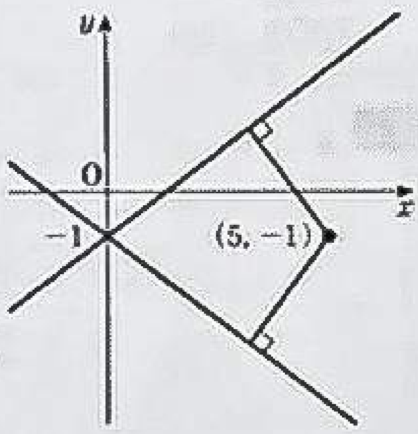
$\therefore\ |5a|=3\sqrt{a^2+16}$

Squaring both sides

$25a^2=9(a^2+16)$

$a^2=9$

$\therefore\ a=\pm3$



4. When k is a constant, find k for which the length of perpendicular dropped from point $(2, 1)$ to line $kx+y+1=0$ is $\sqrt{3}$.

[Sol] $\sqrt{3}=\dfrac{|k\cdot2+1+1|}{\sqrt{k^2+1^2}}$

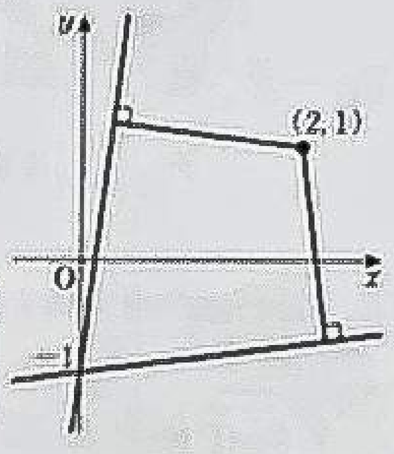
$\therefore\ |2k+2|=\sqrt{3(k^2+1)}$

$(2k+2)^2=3(k^2+1)$

$k^2+8k+1=0$

From Quadratic Formula II (J102), $k=-4\pm\sqrt{4^2-1\cdot1}$

$\therefore\ k=-4\pm\sqrt{15}$



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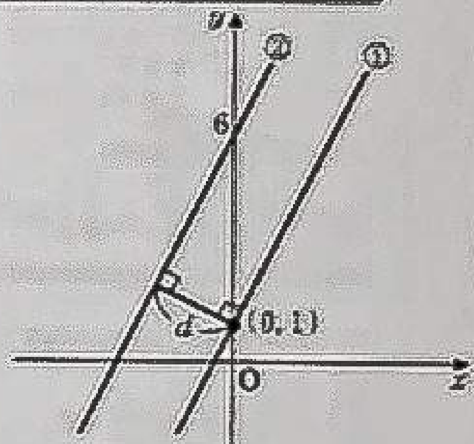
Ex. Find the distance d between two parallel lines $2x-y+1=0$ …① and $2x-y+6=0$ …②.

[Sol] Consider point $(0, 1)$ on ①. ← A point can be chosen either on line ① or ②.

Since the distance between lines ① and ② is equal to the distance from point $(0, 1)$ to line ②,

$$d=\frac{|2\cdot0-1+6|}{\sqrt{2^2+(-1)^2}}=\sqrt{5}$$

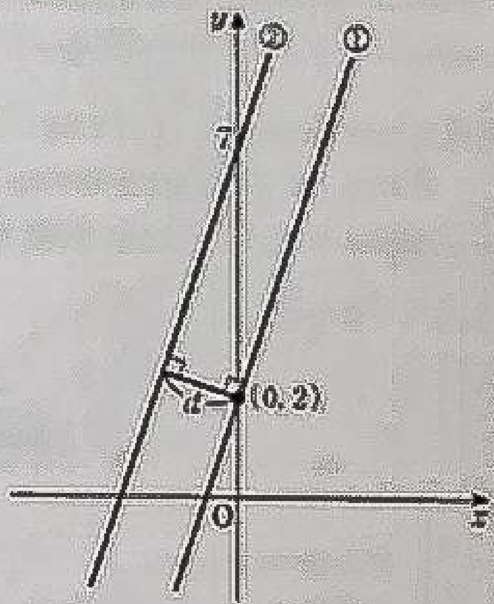


1. Find the distance d between two parallel lines $3x-y+2=0$ …① and $3x-y+7=0$ …②.

[Sol] Consider point $(0, 2)$ on ①.

Since the distance between lines ① and ② is equal to the distance from point $(0, 2)$ to line ②,

$$d=\frac{|3\cdot0-2+7|}{\sqrt{3^2+(-1)^2}}=\frac{5}{\sqrt{10}}=\frac{\sqrt{10}}{2}$$



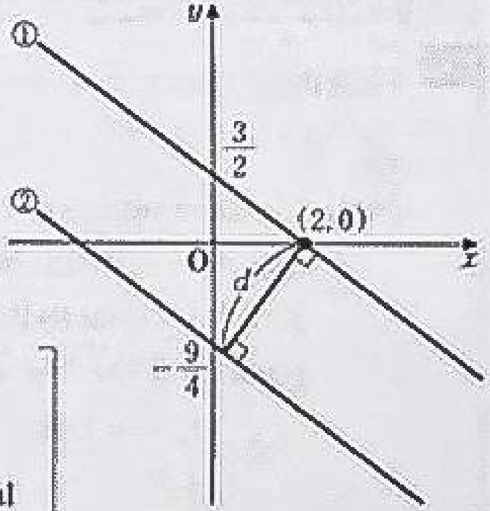
Note The distance between two parallel lines is equal to the length of the perpendicular dropped from an arbitrary point on one line to the other line. The arbitrary point can be any point on the line that is convenient to calculate. For example, a point with integer coordinates, or x-/y-intercept, which contains 0 in the coordinates.

M27b

2. Find the distance d between two parallel lines $3x+4y-6=0$ ···① and $3x+4y+9=0$ ···②.

[Sol] Consider point $(2, 0)$ on ①.
Since the distance between lines ① and ② is equal to the distance from point $(2, 0)$ to line ②,

$$d=\frac{|3\cdot2+4\cdot0+9|}{\sqrt{3^2+4^2}}=3$$



Alternative Solution
Consider point $\left(0, \frac{3}{2}\right)$ on ①.
Since the distance between lines ① and ② is equal to the distance from point $\left(0, \frac{3}{2}\right)$ to line ②,

$$d=\frac{\left|3\cdot0+4\cdot\frac{3}{2}+9\right|}{\sqrt{3^2+4^2}}=3$$

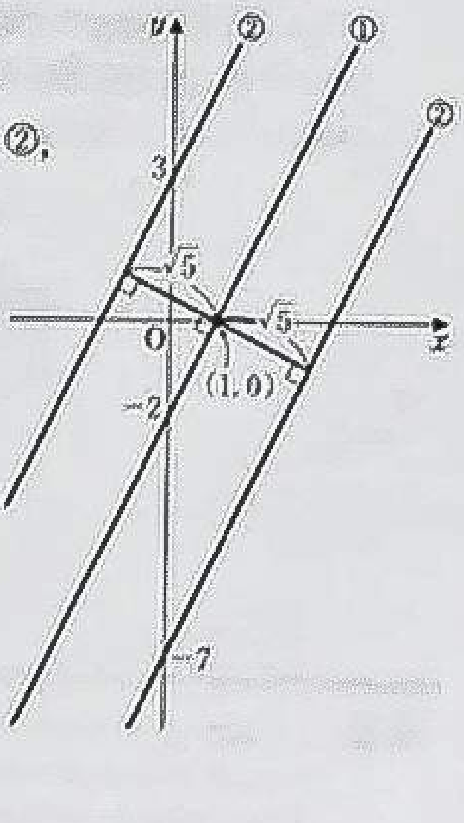
3. Find a for which the distance between two parallel lines $2x-y-2=0$ ···① and $2x-y+a=0$ ···② is $\sqrt{5}$.

[Sol] Consider point $(1, 0)$ on ①.
Since the distance between lines ① and ② is equal to the distance from point $(1, 0)$ to line ②,

$$\sqrt{5}=\frac{|2\cdot1-0+a|}{\sqrt{2^2+(-1)^2}}$$

$5=|a+2|$

$\therefore\ a=3, -7$ From $a+2=\pm5$



Alternative Solution
Consider point $(0, -2)$ on ①.
Since the distance between lines ① and ② is equal to the distance from point $(0, -2)$ to line ②,

$$\sqrt{5}=\frac{|2\cdot0+2+a|}{\sqrt{2^2+(-1)^2}}$$

$5=|a+2|$

$\therefore\ a=3, -7$

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Ex. Find the area S of $\triangle ABC$ with vertices $A(0, -3)$, $B(8, 3)$ and $C(-1, 5)$.

[Sol] The equation of line AB is

$$y+3=\frac{3+3}{8-0}(x-0)$$

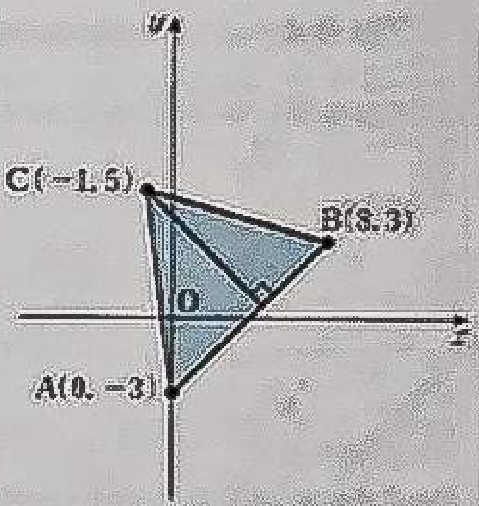
$\therefore\ 3x-4y-12=0\ \cdots①$

The distance d between point C and line ① is

$$d=\frac{|3\cdot(-1)-4\cdot5-12|}{\sqrt{3^2+(-4)^2}}=7$$

Also, $AB=\sqrt{(8-0)^2+(3+3)^2}=10$

$\therefore\ S=\frac{1}{2}\cdot10\cdot7=35$



1. Find the area S of $\triangle ABC$ with vertices $A(-5, -3)$, $B(3, -1)$ and $C(-1, 8)$.

[Sol] The equation of line AB is

$$y+3=\frac{-1+3}{3+5}(x+5)$$

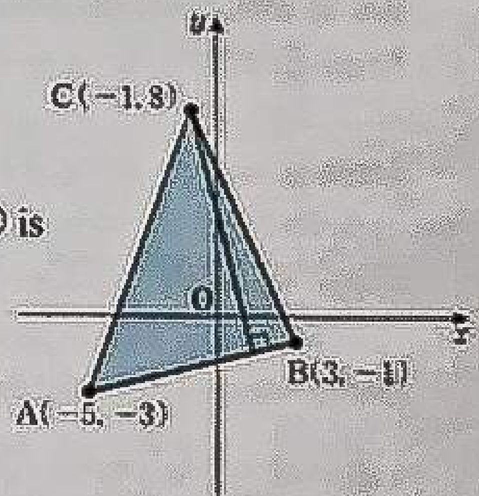
$\therefore\ x-4y-7=0\ \cdots①$

The distance d between point C and line ① is

$$d=\frac{|-1-4\cdot8-7|}{\sqrt{1^2+(-4)^2}}=\frac{40}{\sqrt{17}}=\frac{40\sqrt{17}}{17}$$

Also, $AB=\sqrt{(3+5)^2+(-1+3)^2}=2\sqrt{17}$

$\therefore\ S=\frac{1}{2}\cdot2\sqrt{17}\cdot\frac{40\sqrt{17}}{17}=40$



M28b

2. Find the area S of $\triangle$ABC with vertices A$(-3, -5)$, B$(5, -1)$ and C$(-2, 4)$.

[Sol] The equation of line AB is

$$y+5=\frac{-1+5}{5+3}(x+3)$$

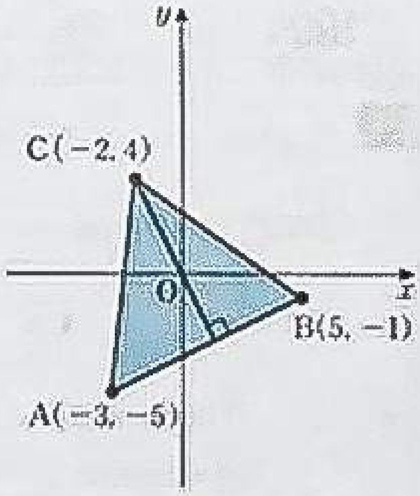
$$\therefore\ x-2y-7=0 \quad \cdots①$$

The distance d between point C and line ① is

$$d=\frac{|-2-2\cdot4-7|}{\sqrt{1^2+(-2)^2}}=\frac{17}{\sqrt{5}}=\frac{17\sqrt{5}}{5}$$

Also, $AB=\sqrt{(5+3)^2+(-1+5)^2}=4\sqrt{5}$

$$\therefore\ S=\frac{1}{2}\cdot4\sqrt{5}\cdot\frac{17\sqrt{5}}{5}=34$$



3. Find the area S of a triangle which is formed by the lines
$2x-y-1=0$ $\cdots①$, $x-4y+3=0$ $\cdots②$ and $3x+2y-19=0$ $\cdots③$.

[Sol] Let A, B and C be the points of intersection of ① and ②, ② and ③, and ③ and ① respectively.

From ① and ②, $x=1$, $y=1$

$\therefore$ A(1, 1)

From ② and ③, $x=5$, $y=2$

$\therefore$ B(5, 2)

From ③ and ①, $x=3$, $y=5$

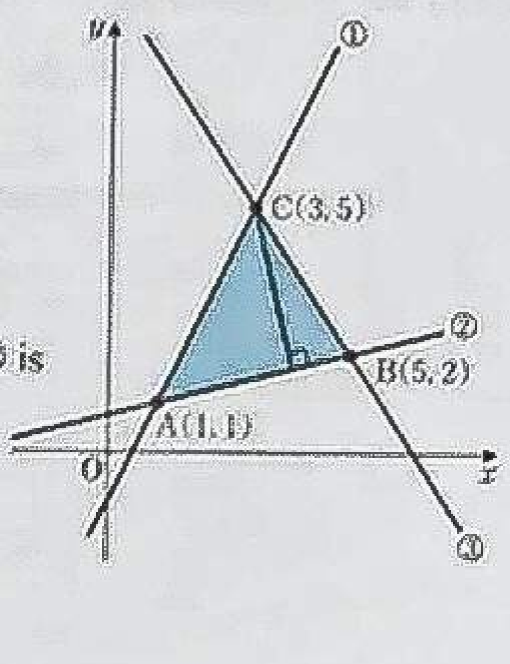
$\therefore$ C(3, 5)

The distance d between point C and line ② is

$$d=\frac{|3-4\cdot5+3|}{\sqrt{1^2+(-4)^2}}=\frac{14}{\sqrt{17}}=\frac{14\sqrt{17}}{17}$$

Also, $AB=\sqrt{(5-1)^2+(2-1)^2}=\sqrt{17}$

$$\therefore\ S=\frac{1}{2}\cdot\sqrt{17}\cdot\frac{14\sqrt{17}}{17}=7$$



Points and Lines 3

Name

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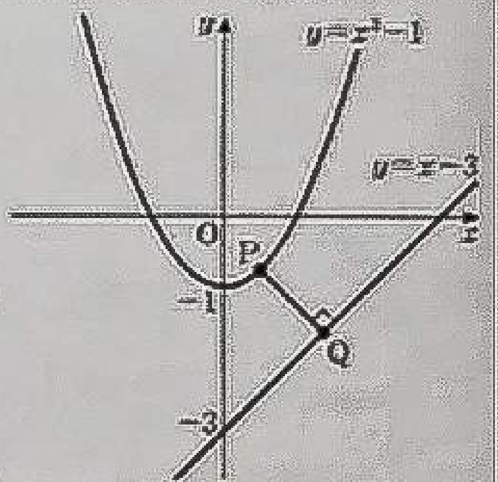
Time : to :

1. Given point P moving along curve $y=x^2-1$ and point Q moving along line $y=x-3$, find the coordinates of point Q and the distance between P and Q when the distance between those two points is minimized.

[Sol] Let point P be $(t,\ t^2-1)$. The distance d between point P and line $y=x-3$, i.e. $x-y-3=0$ ···① is



$$d=\frac{|t-(t^2-1)-3|}{\sqrt{1^2+(-1)^2}}$$
$$=\frac{|-t^2+t-2|}{\sqrt{2}}$$
$$=\frac{|t^2-t+2|}{\sqrt{2}} \quad \boxed{|-A|=|A|}$$
$$=\frac{\left|\left(t-\frac{1}{2}\right)^2+\frac{7}{4}\right|}{\sqrt{2}}$$
$$=\frac{\sqrt{2}}{2}\left(t-\frac{1}{2}\right)^2+\frac{7\sqrt{2}}{8} \quad \boxed{\text{Since } \left(t-\frac{1}{2}\right)^2+\frac{7}{4}>0}$$

Therefore,

when $t=\frac{1}{2}$, the minimum value of d is $\frac{7\sqrt{2}}{8}$.

With this value, point P is $\left(\frac{1}{2},\ -\frac{3}{4}\right)$. ⟵ $\boxed{\text{Substituting } t=\frac{1}{2} \text{ into } (t,\ t^2-1)}$

Since the slope of line $y=x-3$ is 1, the slope of line PQ is -1.

Thus, the equation of line PQ is $y+\frac{3}{4}=-1\cdot\left(x-\frac{1}{2}\right)$.

So, $4x+4y+1=0$ ···②

From ① and ②, $x=\frac{11}{8}$, $y=-\frac{13}{8}$

$\therefore$ Q$\left(\frac{11}{8},\ -\frac{13}{8}\right)$

M29b

2. Given points A$(0, 1)$, B$(2, 5)$ and point P moving along parabola $y=x^2+4x+7$ ⋯①, find the minimum value of area S of $\triangle$PAB.

[Sol] The equation of line AB is

$$y-1=\frac{5-1}{2-0}(x-0)$$

$\therefore\ 2x-y+1=0$ ⋯②

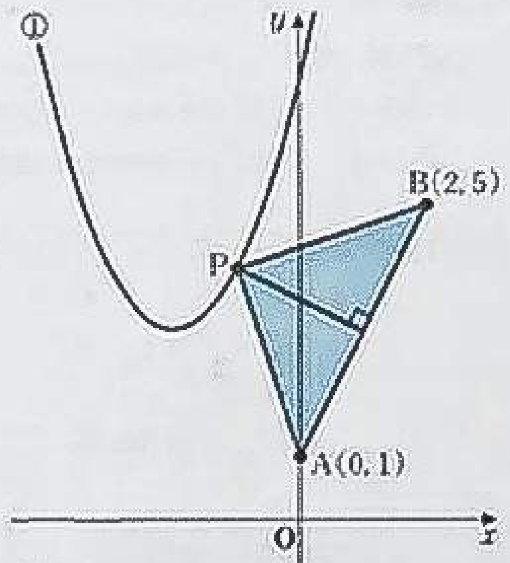
Let point P be (t, t^2+4t+7). The distance d between point P and line ② is

$$d=\frac{|2\cdot t-(t^2+4t+7)+1|}{\sqrt{2^2+(-1)^2}}$$

$$=\frac{|-t^2-2t-6|}{\sqrt{5}}$$

$$=\frac{|t^2+2t+6|}{\sqrt{5}}$$

$\boxed{|-A|=|A|}$



Also, $AB=\sqrt{(2-0)^2+(5-1)^2}=2\sqrt{5}$

$$\therefore\ S=\frac{1}{2}\cdot 2\sqrt{5}\cdot\frac{|t^2+2t+6|}{\sqrt{5}}$$

$$=|t^2+2t+6|$$

$$=|(t+1)^2+5|$$

$$=(t+1)^2+5$$

Since $(t+1)^2+5>0$

Therefore,
when $t=-1$, the minimum value of S is 5.

Alternative Solution
The equation of line AB is
$2x-y+1=0$ ⋯②
Let point P be (t, t^2+4t+7).
The distance d between point P and line ② is

$$d=\frac{|t^2+2t+6|}{\sqrt{5}}$$

$$=\frac{|(t+1)^2+5|}{\sqrt{5}}$$

$$=\frac{\sqrt{5}}{5}(t+1)^2+\sqrt{5}$$

Therefore,
when $t=-1$, the minimum value of d is $\sqrt{5}$.
Also,
$AB=\sqrt{(2-0)^2+(5-1)^2}=2\sqrt{5}$

$$\therefore\ S=\frac{1}{2}\cdot 2\sqrt{5}\cdot\sqrt{5}=5$$

Thus,
when $t=-1$, the minimum value of S is 5.

Points and Lines 3

Name

Date / /

Time : to :

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1. Given $\triangle ABC$ with vertices $A(3, 4)$, $B(0, -8)$ and $C(8, 0)$, let AP and BQ be the perpendiculars dropped from A and B to their opposite sides of $\triangle ABC$. Find the coordinates of orthocenter H of $\triangle ABC$. ➡ M22

[Sol] Since the slope of line BC is $\frac{0+8}{8-0}=1$,

the slope of perpendicular AP is -1.

Therefore, the equation of perpendicular AP is $y-4=-1\cdot(x-3)$.

So, $y=-x+7$ …①

Since the slope of line AC is $\frac{0-4}{8-3}=-\frac{4}{5}$,

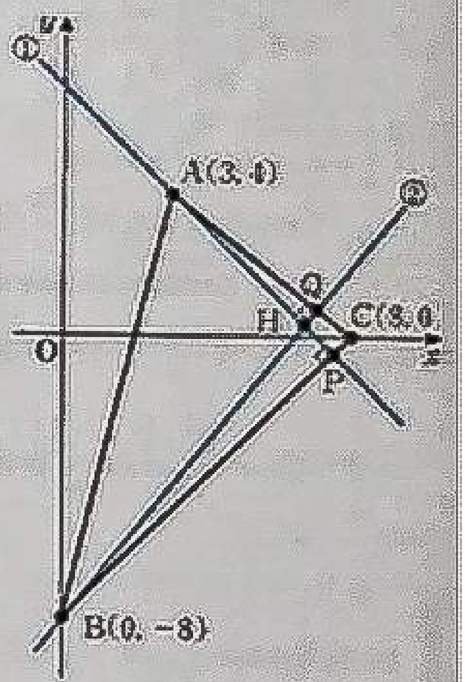
the slope of perpendicular BQ is $\frac{5}{4}$.

Therefore, the equation of perpendicular BQ is $y+8=\frac{5}{4}(x-0)$.

So, $y=\frac{5}{4}x-8$ …②

From ① and ②, $x=\frac{20}{3}$, $y=\frac{1}{3}$

$\therefore H\left(\frac{20}{3}, \frac{1}{3}\right)$



2. Find the coordinates of point P which is on the x-axis and at a distance of 2 units from line $3x-4y+1=0$. ➡ M26

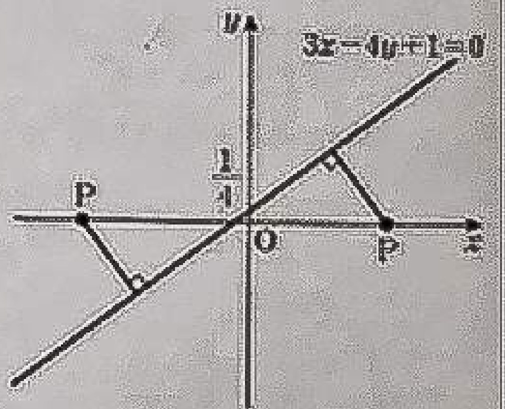
[Sol] Let P be $P(p, 0)$.

$$2=\frac{|3\cdot p-4\cdot 0+1|}{\sqrt{3^2+(-4)^2}}$$

$\therefore |3p+1|=10$

$p=3, -\frac{11}{3}$

$\therefore P(3, 0), \left(-\frac{11}{3}, 0\right)$



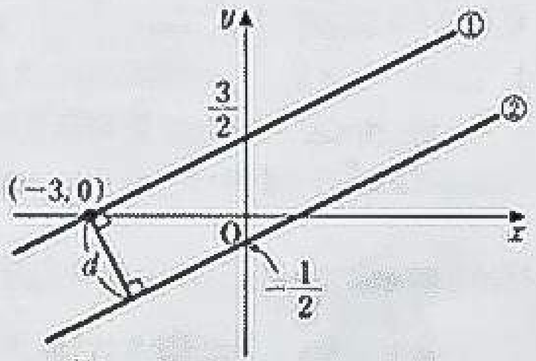
M30b

3. Find the distance d between two parallel lines $x-2y+3=0$ ⋯① and $x-2y-1=0$ ⋯②.

➡ M27

[Sol] Consider point $(-3, 0)$ on ①.
Since the distance between lines ① and ② is equal to the distance from point $(-3, 0)$ to line ②,

$$d=\frac{|-3-2\cdot0-1|}{\sqrt{1^2+(-2)^2}}=\frac{4}{\sqrt{5}}=\frac{4\sqrt{5}}{5}$$



Alternative Solution
Consider point $\left(0, \frac{3}{2}\right)$ on ①.
Since the distance between lines ① and ② is equal to the distance from point $\left(0, \frac{3}{2}\right)$ to line ②,

$$d=\frac{\left|0-2\cdot\frac{3}{2}-1\right|}{\sqrt{1^2+(-2)^2}}=\frac{4}{\sqrt{5}}=\frac{4\sqrt{5}}{5}$$

4. Find the area S of $\triangle$ABC with vertices A$(-1, 3)$, B$(5, -2)$ and C$(2, 4)$.

➡ M28

[Sol] The equation of line AB is

$$y-3=\frac{-2-3}{5+1}(x+1)$$

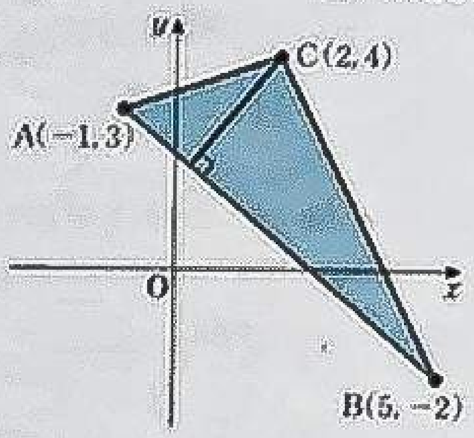
$\therefore\ 5x+6y-13=0$ ⋯①

The distance d between point C and line ① is

$$d=\frac{|5\cdot2+6\cdot4-13|}{\sqrt{5^2+6^2}}=\frac{21}{\sqrt{61}}=\frac{21\sqrt{61}}{61}$$

Also, $AB=\sqrt{(5+1)^2+(-2-3)^2}=\sqrt{61}$

$$\therefore\ S=\frac{1}{2}\cdot\sqrt{61}\cdot\frac{21\sqrt{61}}{61}=\frac{21}{2}$$



Circles 1

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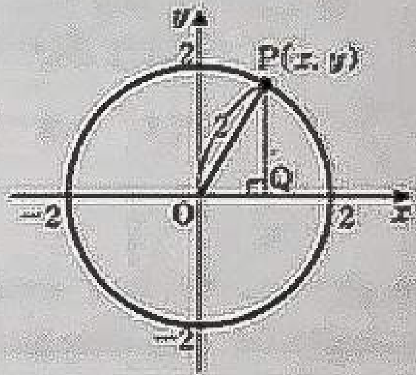
As shown in the diagram, let $P(x, y)$ be a point at a fixed distance of 2 units from origin O. Find the equation of point P.

[Sol] Let a perpendicular dropped from point P to the x-axis be PQ.

In $\triangle OPQ$,

$OQ^2+PQ^2=OP^2$ ⟵ The Pythagorean Theorem

$\therefore \boxed{x^2+y^2=4}$



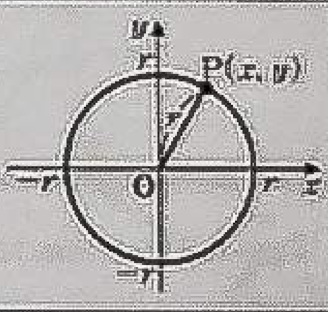
Answer: $x^2+y^2=4$

The equation of point P above is the circle with center at origin O and radius 2. Generally, the following formula is true.

Equation of a Circle I

The equation of a circle with center at origin O and radius r is

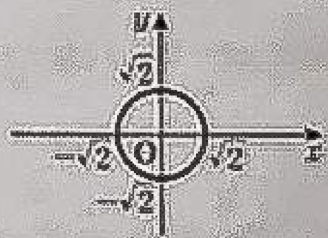
$$x^2+y^2=r^2$$



1. Find the equation of each given circle.

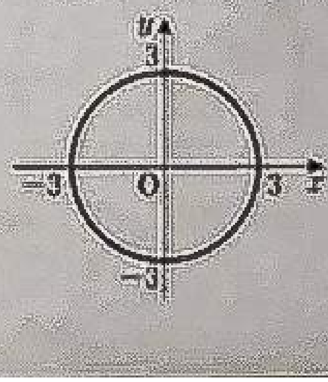
(1) The circle with center at origin O and radius $\sqrt{2}$.

[Sol] $x^2+y^2=2$



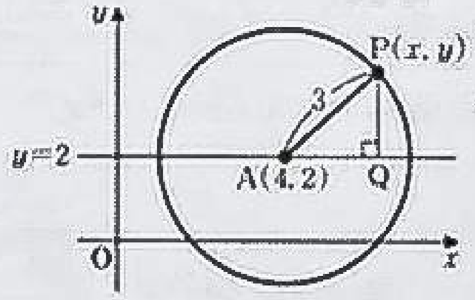
(2) The circle with center at origin O and radius 3.

[Sol] $x^2+y^2=9$



M31b

As shown in the diagram, let P(x, y) be a point at a fixed distance of 3 units from point A$(4, 2)$. Find the equation of point P.



[Sol] Let a perpendicular from point P to line $y=2$ be PQ.

In $\triangle$APQ,

$$AQ^2+PQ^2=AP^2$$

$$\therefore \boxed{(x-4)^2+(y-2)^2=9}$$

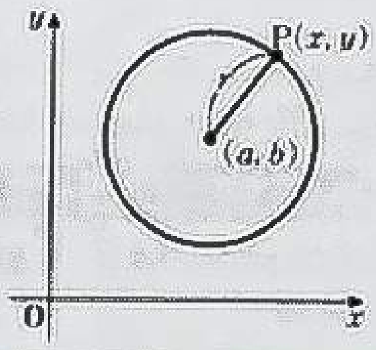
Answer: $(x-4)^2+(y-2)^2=9$

The equation of point P above is the circle with center at point $(4, 2)$ and radius 3. Generally, the following formula is true.

Equation of a Circle II

The equation of a circle with center at point (a, b) and radius r is

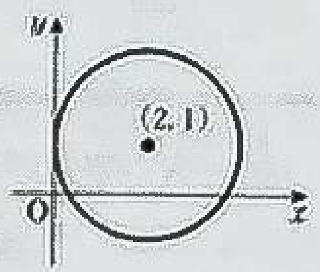
$$(x-a)^2+(y-b)^2=r^2$$



2. Find the equation of each given circle.

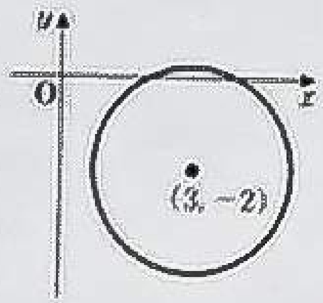
(1) The circle with center at point $(2, 1)$ and radius 2.

[Sol] $(x-2)^2+(y-1)^2=4$



(2) The circle with center at point $(3, -2)$ and radius $\sqrt{5}$.

[Sol] $(x-3)^2+(y+2)^2=5$



M32a

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Circles 1

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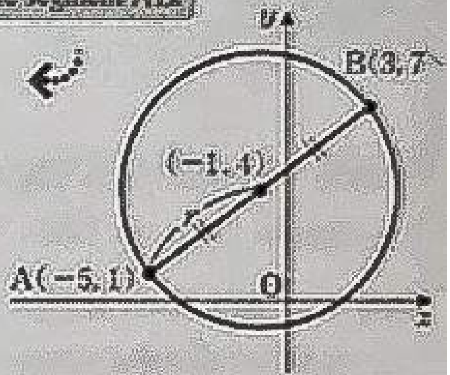
1. Find the equation of the circle with points A$(-5, 1)$ and B$(3, 7)$ at the endpoints of its diameter.

[Sol] The center of the circle is

$\left(\boxed{\dfrac{-5+3}{2}}, \boxed{\dfrac{1+7}{2}}\right)$, i.e. $(\boxed{-1}, \boxed{4})$. ← The center of a circle is the midpoint of line segment AB.



Let the radius of the circle be r.

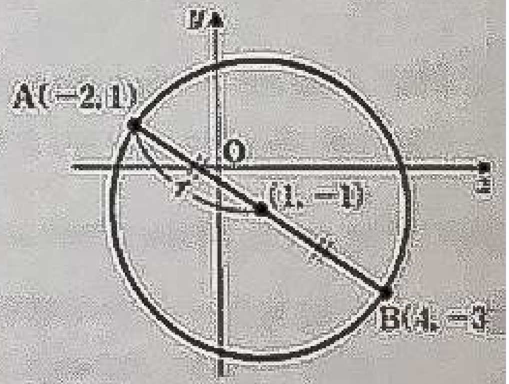
$r=\sqrt{(-1+5)^2+(4-1)^2}=5$

$\therefore\ (x+1)^2+(y-4)^2=25$

2. Find the equation of the circle with points A$(-2, 1)$ and B$(4, -3)$ at the endpoints of its diameter.

[Sol] The center of the circle is

$\left(\dfrac{-2+4}{2}, \dfrac{1-3}{2}\right)$, i.e. $(1, -1)$.



Let the radius of the circle be r.

$r=\sqrt{(1+2)^2+(-1-1)^2}=\sqrt{13}$

$\therefore\ (x-1)^2+(y+1)^2=13$

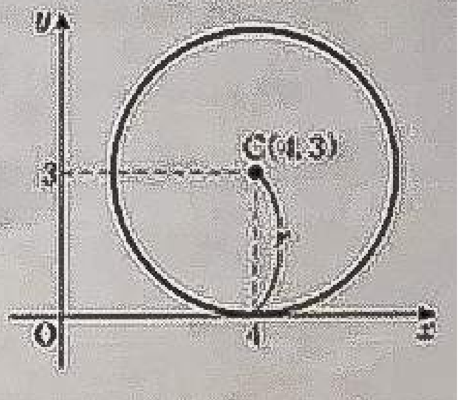
3. Find the equation of the circle with center at point C$(4, 3)$ and touching the x-axis at only one point.

[Sol] Let the radius of the circle be r.

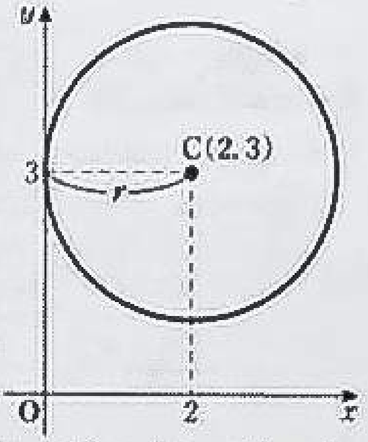
r is the distance between point $(4, 3)$ and the x-axis.

$\therefore\ r=3$ ← From the diagram

$\therefore\ (x-4)^2+(y-3)^2=9$



4. Find the equation of the circle with center at point C(2, 3) and touching the y-axis at only one point.



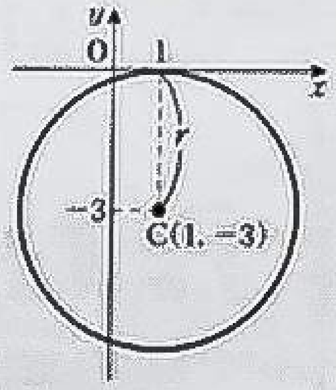
[Sol] Let the radius of the circle be r.

r is the distance between point (2, 3) and the y-axis.

$\therefore\ r=2$

$\therefore\ (x-2)^2+(y-3)^2=4$

5. Find the equation of the circle with center at point C(1, −3) and touching the x-axis at only one point.



[Sol] Let the radius of the circle be r.

r is the distance between point (1, −3) and the x-axis.

$\therefore\ r=3$

$\therefore\ (x-1)^2+(y+3)^2=9$

6. Find a for the circle with points A(1, 1) and B(−3, a) at the endpoints of its diameter and touching the x-axis at only one point.

[Sol] The center of the circle is $\left(\frac{1-3}{2},\ \frac{1+a}{2}\right)$, i.e. $\left(-1,\ \frac{a+1}{2}\right)$.

Let the radius of the circle be r.

r is the distance between the center at point $\left(-1,\ \frac{a+1}{2}\right)$ and the x-axis.

$\therefore\ r=\frac{|a+1|}{2}$ ← Since r is the distance between the center and the x-axis, it is equal to the absolute value of the y-coordinate.

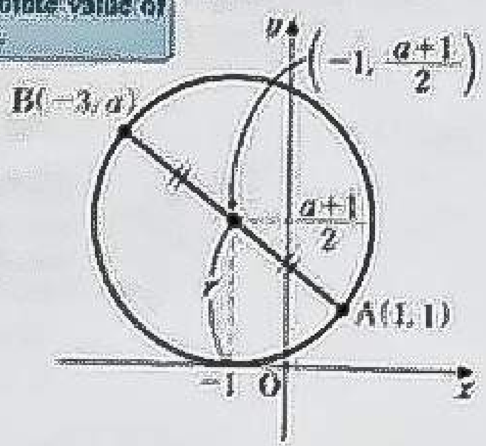
Therefore, the equation of the circle is

$$(x+1)^2+\left(y-\frac{a+1}{2}\right)^2=\left(\frac{a+1}{2}\right)^2$$

Since this passes through point A(1, 1),

$$(1+1)^2+\left(1-\frac{a+1}{2}\right)^2=\left(\frac{a+1}{2}\right)^2$$

$\therefore\ a=4$



Circles 1

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Rewrite each given equation in the form $(x-a)^2+(y-b)^2=r^2$ and state the center and radius of the circle.

Ex. $x^2+y^2-2x-4y-20=0$

$(x^2-2x)+(y^2-4y)=20$

$(x^2-2x+1^2)+(y^2-4y+2^2)=20+1^2+2^2$

[Sol] $(x-1)^2+(y-2)^2=5^2$

$\therefore$ center: (1, 2), radius: 5

(1) $x^2+y^2-6x-2y-6=0$

[Sol] $(x-3)^2+(y-1)^2=4^2$

$\therefore$ **center: (3, 1), radius: 4**

(2) $x^2+y^2-8x+6y+16=0$

[Sol] $(x-4)^2+(y+3)^2=3^2$

$\therefore$ **center: (4, −3), radius: 3**

(3) $x^2+y^2+4x+8y+15=0$

[Sol] $(x+2)^2+(y+4)^2=(\sqrt{5})^2$

$\therefore$ **center: (−2, −4), radius: $\sqrt{5}$**

M33b

(4) $x^2+y^2+2x-3y+3=0$

[Sol] $(x+1)^2+\left(y-\frac{3}{2}\right)^2=\left(\frac{1}{2}\right)^2$

$\therefore$ **center: $\left(-1, \frac{3}{2}\right)$, radius: $\frac{1}{2}$**

(5) $x^2+y^2+\frac{3}{2}x+\frac{1}{2}=0$

[Sol] $\left(x+\frac{3}{4}\right)^2+y^2=\left(\frac{1}{4}\right)^2$

$\therefore$ **center: $\left(-\frac{3}{4}, 0\right)$, radius: $\frac{1}{4}$**

(6) $x^2+y^2-\sqrt{3}x+y-2=0$

[Sol] $\left(x-\frac{\sqrt{3}}{2}\right)^2+\left(y+\frac{1}{2}\right)^2=(\sqrt{3})^2$

$\therefore$ **center: $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, radius: $\sqrt{3}$**

(7) $3x^2+3y^2+10y=0$

[Sol] $x^2+y^2+\frac{10}{3}y=0$

$x^2+\left(y+\frac{5}{3}\right)^2=\left(\frac{5}{3}\right)^2$

$\therefore$ **center: $\left(0, -\frac{5}{3}\right)$, radius: $\frac{5}{3}$**

Circles 1

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Time : to :

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Ex. Find the equation of the circle passing through three points $A(1, -1)$, $B(5, 1)$ and $C(-2, 0)$.

[Sol] Let $x^2+y^2+ax+by+c=0$.

Since this circle passes through points A, B and C,

$a-b+c=-2 \quad \cdots①$ ⟵ $1^2+(-1)^2+a\cdot1+b\cdot(-1)+c=0$

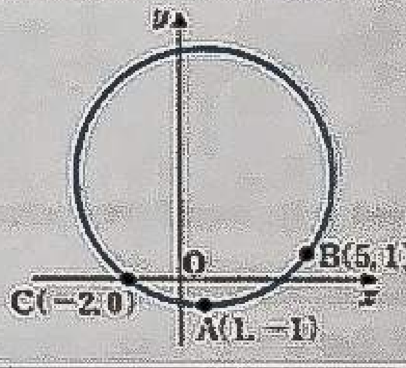
$5a+b+c=-26 \quad \cdots②$ ⟵ $5^2+1^2+a\cdot5+b\cdot1+c=0$

$-2a+c=-4 \quad \cdots③$ ⟵ $(-2)^2+0^2+a\cdot(-2)+b\cdot0+c=0$

From ①~③,

$a=-2,\ b=-8,\ c=-8$

$\therefore\ x^2+y^2-2x-8y-8=0$



1. Find the equation of the circle passing through three points $A(3, 5)$, $B(2, -2)$ and $C(-6, 2)$.

[Sol] Let $x^2+y^2+ax+by+c=0$.

Since this circle passes through points A, B and C,

$3a+5b+c=-34 \quad \cdots①$

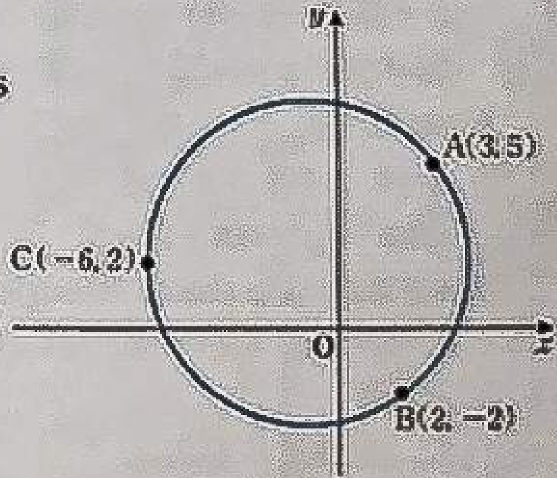
$2a-2b+c=-8 \quad \cdots②$

$-6a+2b+c=-40 \quad \cdots③$

From ①~③,

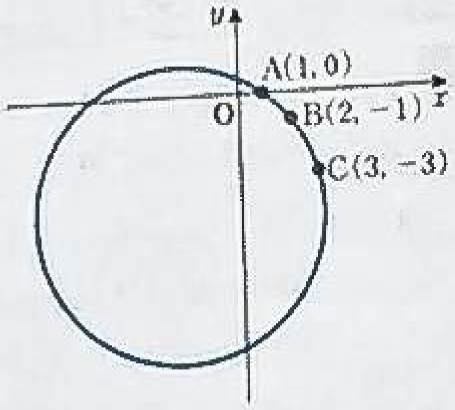
$a=2,\ b=-4,\ c=-20$

$\therefore\ x^2+y^2+2x-4y-20=0$



M34b

2. Find the equation of the circle passing through three points A$(1, 0)$, B$(2, -1)$ and C$(3, -3)$.



[Sol] Let $x^2+y^2+ax+by+c=0$.
Since this circle passes through points A, B and C,

$a+c=-1$ $\cdots$①

$2a-b+c=-5$ $\cdots$②

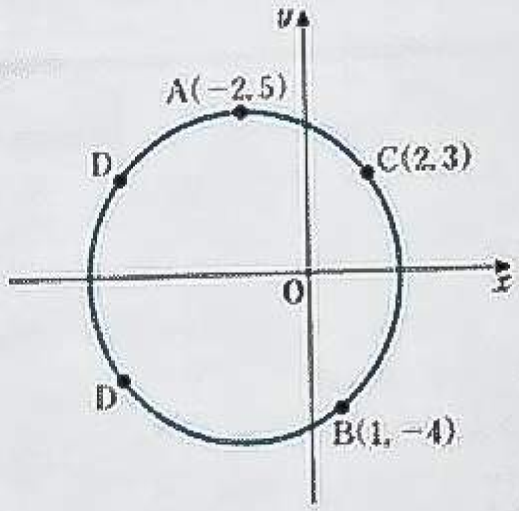
$3a-3b+c=-18$ $\cdots$③

From ①~③,

$a=5,\ b=9,\ c=-6$

$\therefore\ x^2+y^2+5x+9y-6=0$

3. Find p for which four points A$(-2, 5)$, B$(1, -4)$, C$(2, 3)$ and D$(-6, p)$ all lie on the same circle.



[Sol] Let $x^2+y^2+ax+by+c=0$.
Since this circle passes through points A, B and C,

$-2a+5b+c=-29$ $\cdots$①

$a-4b+c=-17$ $\cdots$②

$2a+3b+c=-13$ $\cdots$③

From ①~③,

$a=4,\ b=0,\ c=-21$

$\therefore\ x^2+y^2+4x-21=0$

Since this circle passes through point D,

$(-6)^2+p^2+4\cdot(-6)-21=0$

$p^2=9$

$\therefore\ p=\pm 3$

A circle passing through all three vertices of $\triangle$ABC is called a ***circumscribed circle*** of $\triangle$ABC. The center of a circumscribed circle coincides with the circumcenter of the triangle.

Circles 1

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Ex. Find the equation of the circle passing through two points $A(-2, -3)$ and $B(4, 5)$ with its center on line $y=2x-1$.

[Sol] Let the center of the circle be $(a, 2a-1)$. ← Since the center of the circle lies on $y=2x-1$

Let the radius of the circle be r.

$(x-a)^2+[y-(2a-1)]^2=r^2$

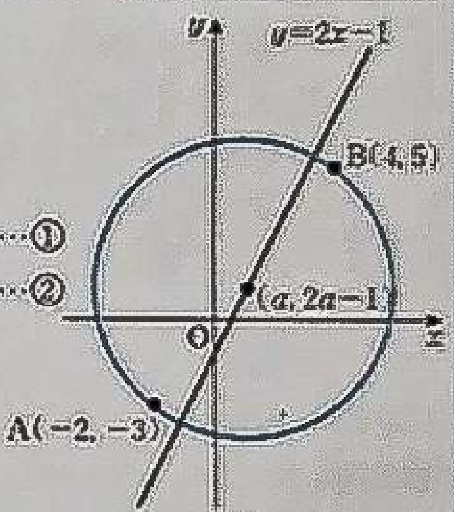
Since this circle passes through two points A and B,

$(-2-a)^2+[-3-(2a-1)]^2=r^2 \quad \cdots ①$

$(4-a)^2+[5-(2a-1)]^2=r^2 \quad \cdots ②$

From ① and ②, $a=1, r^2=25$

$\therefore\ (x-1)^2+(y-1)^2=25$



1. Find the equation of the circle passing through two points $A(-2, 5)$ and $B(4, -3)$ with its center on line $y=x-1$.

[Sol] Let the center of the circle be $(a, a-1)$.

Let the radius of the circle be r.

$(x-a)^2+[y-(a-1)]^2=r^2$

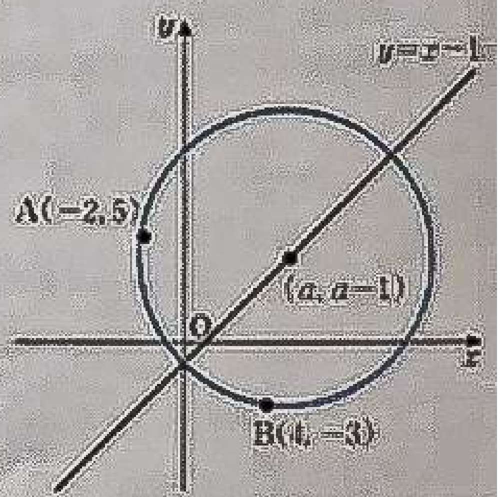
Since this circle passes through two points A and B,

$(-2-a)^2+[5-(a-1)]^2=r^2 \quad \cdots ①$

$(4-a)^2+[-3-(a-1)]^2=r^2 \quad \cdots ②$

From ① and ②, $a=5, r^2=50$

$\therefore\ (x-5)^2+(y-4)^2=50$



2. Find the equation of the circle passing through two points $A(4, 1)$ and $B(3, 2)$ with its center on the x-axis.

Since the center of the circle lies on the x-axis

[Sol] Let the center of the circle be $(a, 0)$.

Let the radius of the circle be r.

$$(x-a)^2+y^2=r^2$$

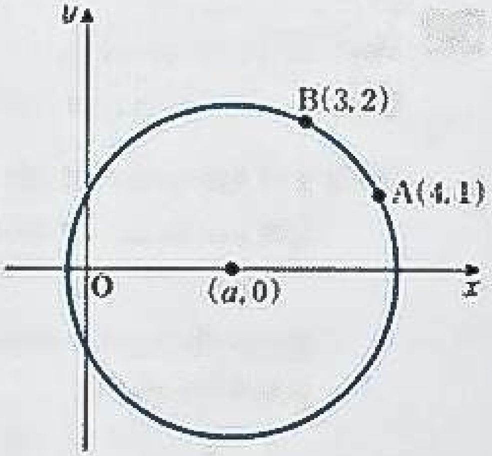
Since this circle passes through two points A and B,

$$(4-a)^2+1^2=r^2 \cdots ①$$

$$(3-a)^2+2^2=r^2 \cdots ②$$

From ① and ②, $a=2$, $r^2=5$

$$\therefore (x-2)^2+y^2=5$$



3. Find the equation of the circle with its center on line $y=x+2$ which passes through point $A(6, 1)$ and touches the x-axis at only one point.

[Sol] Let the center of the circle be $(a, a+2)$.

Let the radius of the circle be r.

$$(x-a)^2+[y-(a+2)]^2=r^2$$

Since this circle passes through point A,

$$(6-a)^2+[1-(a+2)]^2=r^2 \cdots ①$$

Also, since this circle touches the x-axis at only one point,

$$r=|a+2| \cdots ②$$

Since r is the distance between the center and the x-axis, it is equal to the absolute value of the y-coordinate.

From ① and ②,

$$(6-a)^2+[1-(a+2)]^2=(a+2)^2$$

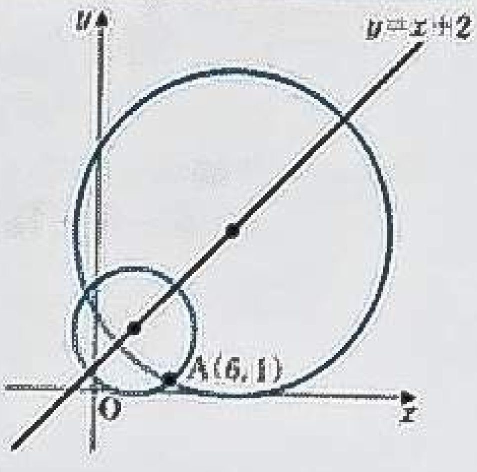
$$a^2-14a+33=0$$

$$(a-3)(a-11)=0$$

$$\therefore a=3, 11$$

$$\therefore (x-3)^2+(y-5)^2=25$$

$$(x-11)^2+(y-13)^2=169$$



Circles 1

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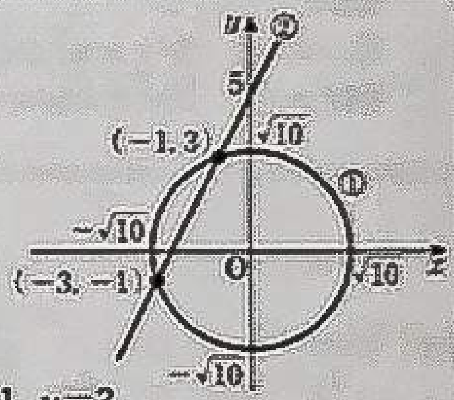
Find the positional relationships between the following circles and lines (for example, "they intersect at two different points," "they are tangent and intersect at only one point" or "they do not intersect"), and find the coordinates of the common point(s) if there are any.

Ex. [1] $\begin{cases} x^2+y^2=10 & \cdots① \\ y=2x+5 & \cdots② \end{cases}$



[Sol] From ① and ②,

$x^2+4x+3=0$ ← Substituting ② into ① and simplifying

$(x+3)(x+1)=0$

$x=-3,\ -1$

Substituting into ②,

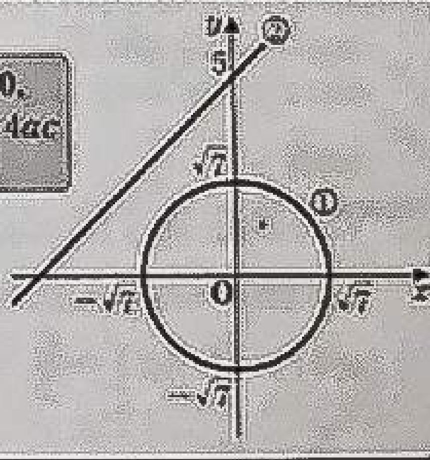
when $x=-3$, $y=-1$ and when $x=-1$, $y=3$

Therefore, they intersect at two different points.

The common points are $(-3,\ -1)$ and $(-1,\ 3)$.

[2] $\begin{cases} x^2+y^2=7 & \cdots① \\ y=x+5 & \cdots② \end{cases}$

When $ax^2+bx+c=0$, discriminant $D=b^2-4ac$ (K76)



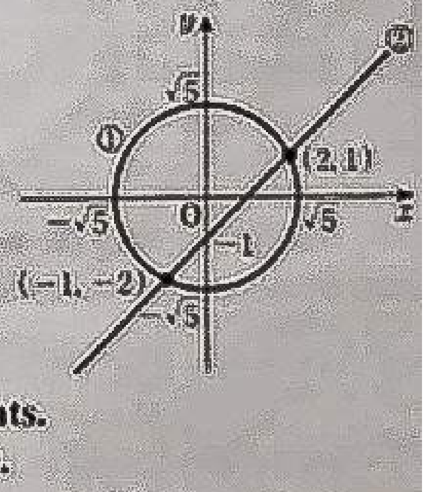
[Sol] From ① and ②,

$x^2+5x+9=0$

$D=5^2-4\cdot1\cdot9=-11<0$

Therefore, they do not intersect.

(1) $\begin{cases} x^2+y^2=5 & \cdots① \\ y=x-1 & \cdots② \end{cases}$



[Sol] From ① and ②,

$x^2-x-2=0$

$(x+1)(x-2)=0$

$x=-1,\ 2$

Substituting into ②,

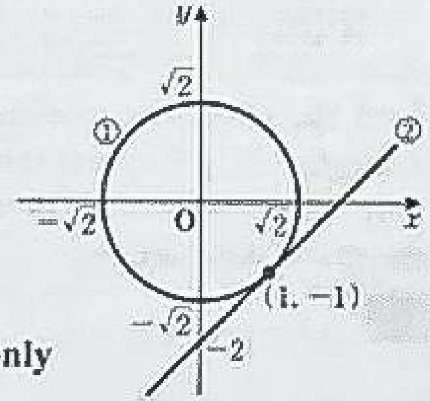
when $x=-1$, $y=-2$ and when $x=2$, $y=1$

Therefore, they intersect at two different points.

The common points are $(-1,\ -2)$ and $(2,\ 1)$.

M36b

(2) $\begin{cases} x^2+y^2=2 & \cdots① \\ y=x-2 & \cdots② \end{cases}$



[Sol] From ① and ②,

$x^2-2x+1=0$

$(x-1)^2=0$

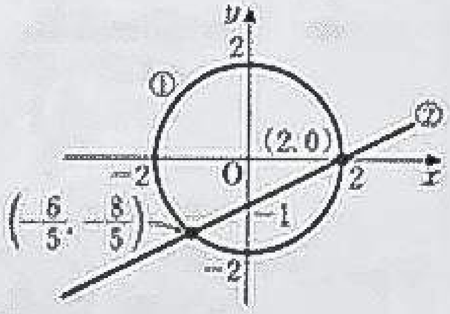
$x=1$

Substituting into ②,

when $x=1$, $y=-1$

Therefore, they are tangent and intersect at only one point. The common point is $(1, -1)$.

(3) $\begin{cases} x^2+y^2=4 & \cdots① \\ x-2y=2 & \cdots② \end{cases}$



[Sol] From ① and ②,

$5y^2+8y=0$ ← From ②, $x=2y+2$ Substituting it into ①

$y(5y+8)=0$

$y=0,\ -\frac{8}{5}$

Substituting into ②,

when $y=0$, $x=2$ and

when $y=-\frac{8}{5}$, $x=-\frac{6}{5}$

Therefore, they intersect at two different points. The common points are $(2, 0)$ and $\left(-\frac{6}{5}, -\frac{8}{5}\right)$.

Alternative Solution

From ① and ②,

$5x^2-4x-12=0$ ← From ②, $y=\frac{1}{2}x-1$ Substituting it into ①

$(5x+6)(x-2)=0$

$x=-\frac{6}{5},\ 2$

Substituting into ①,

when $x=-\frac{6}{5}$, $y=-\frac{8}{5}$ and

when $x=2$, $y=0$

Therefore,

they intersect at two different points. The common points are $\left(-\frac{6}{5}, -\frac{8}{5}\right)$ and $(2, 0)$.

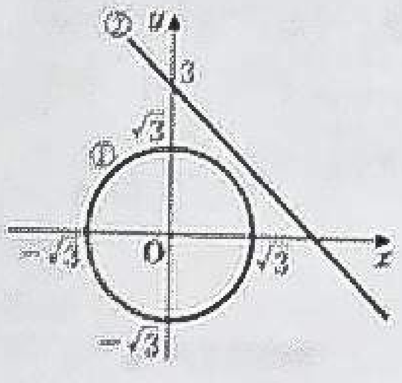
(4) $\begin{cases} x^2+y^2=3 & \cdots① \\ y=-x+3 & \cdots② \end{cases}$

[Sol] From ① and ②,

$x^2-3x+3=0$

$D=(-3)^2-4\cdot1\cdot3=-3<0$

Therefore, they do not intersect.



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Positional Relationship between a Line and a Circle

When the quadratic equation $ax^2+bx+c=0$ is obtained after y is eliminated from each equation of a line and a circle, let the discriminant be $D(=b^2-4ac)$.

$D>0 \Leftrightarrow$ They intersect at two different points.

$D=0 \Leftrightarrow$ They are tangent and intersect at only one point.

$D<0 \Leftrightarrow$ They do not intersect.

Ex. Find the range of values of k for which circle $x^2+y^2=2$ …① intersects with line $y=-x+k$ …② at two different points.

[Sol] From ① and ②,

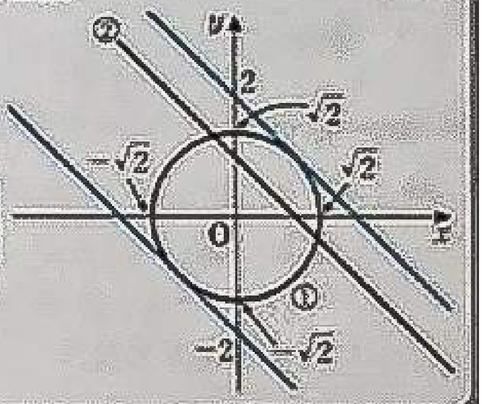
$$2x^2-2kx+k^2-2=0 \quad \boxed{\text{Substituting ② into ①}}$$

$$\frac{D}{4}=(-k)^2-2(k^2-2)$$

$$=-k^2+4>0$$

$$\therefore\ k^2<4$$

$$\therefore\ -2<k<2$$



1. Find the range of values of k for which circle $x^2+y^2=1$ …① intersects with line $y=x+k$ …② at two different points.

[Sol] From ① and ②,

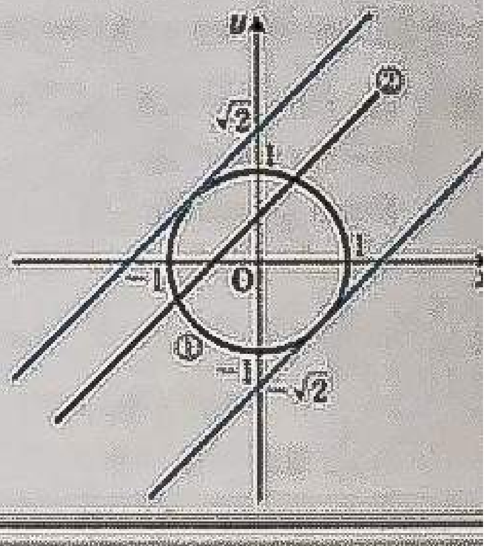
$$2x^2+2kx+k^2-1=0$$

$$\frac{D}{4}=k^2-2(k^2-1)$$

$$=-k^2+2>0$$

$$\therefore\ k^2<2$$

$$\therefore\ -\sqrt{2}<k<\sqrt{2}$$

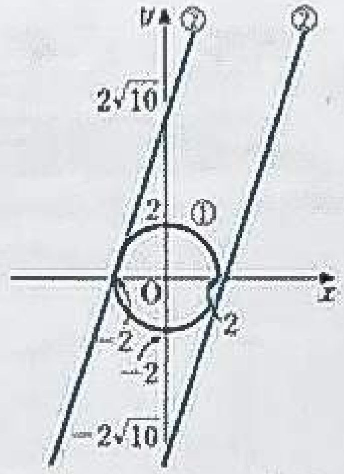


Note Summary

When a quadratic equation is shown in the form $ax^2+2b'x+c=0$, use $\frac{D}{4}=b'^2-ac$.

M37b

2. Find the value of k for which circle $x^2+y^2=4$ ⋯① is tangent to line $y=3x+k$ ⋯②.



[Sol] From ① and ②,

$$10x^2+6kx+k^2-4=0$$

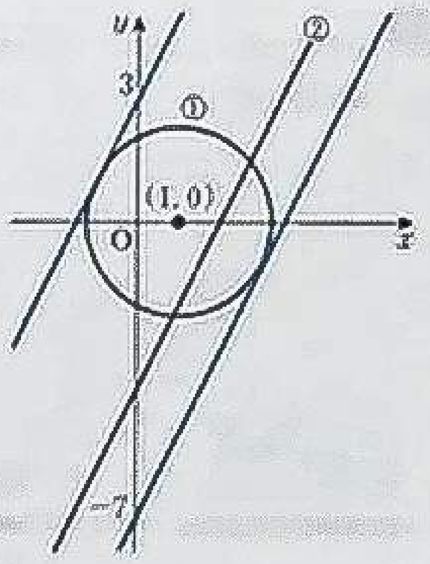
$$\frac{D}{4}=(3k)^2-10(k^2-4)$$

$$=-k^2+40=0$$

$$\therefore\ k^2=40$$

$$\therefore\ k=\pm 2\sqrt{10}$$

3. Find the range of values of k for which circle $x^2+y^2-2x=4$ ⋯① has common points with line $y=2x+k$ ⋯②.



[Sol] From ① and ②,

$$5x^2+2(2k-1)x+k^2-4=0$$

$$\frac{D}{4}=(2k-1)^2-5(k^2-4)$$

$$=-k^2-4k+21\geqq 0$$

$$\therefore\ k^2+4k-21\leqq 0$$

$$(k+7)(k-3)\leqq 0$$

$$\therefore\ -7\leqq k\leqq 3$$

When with common points, $\frac{D}{4}\geqq 0$

4. Find the range of values of a for which circle $(x-2)^2+(y-2)^2=1$ intersects with line $y=ax+1$ at two different points.

[Sol] Let $(x-2)^2+(y-2)^2=1$ ⋯① and $y=ax+1$ ⋯②.

From ① and ②,

$$(a^2+1)x^2-2(a+2)x+4=0$$

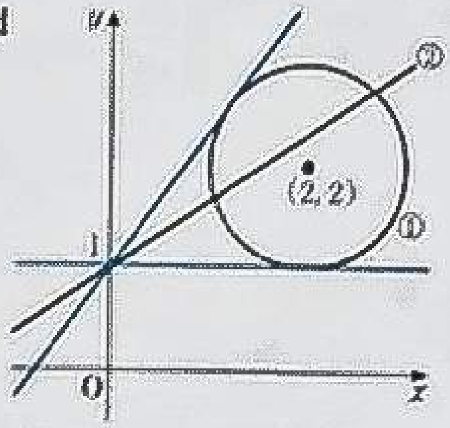
$$\frac{D}{4}=[-(a+2)]^2-(a^2+1)\cdot 4$$

$$=-3a^2+4a>0$$

$$\therefore\ 3a^2-4a<0$$

$$a(3a-4)<0$$

$$\therefore\ 0<a<\frac{4}{3}$$



Circles 1

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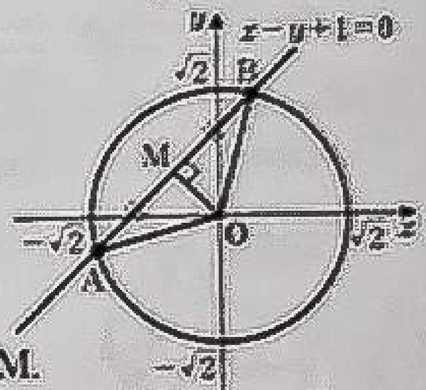
Ex. Let the points of intersection of circle $x^2+y^2=2$ and line $x-y+1=0$ be A and B. Find the length of line segment AB.



[Sol] The distance d from center $(0, 0)$ to line $x-y+1=0$ is

$$d=\frac{|0-0+1|}{\sqrt{1^2+(-1)^2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

Also, the radius is $\sqrt{2}$.

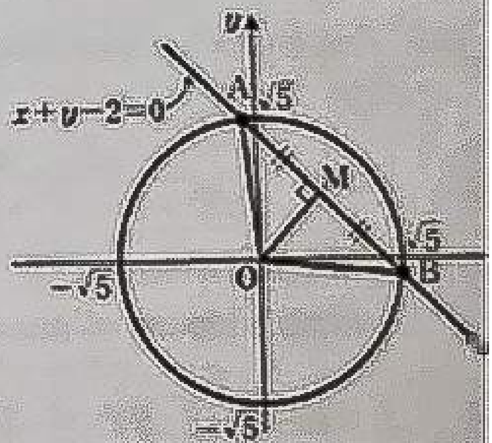
Let the midpoint of line segment AB be M.

$$\mathrm{AM}=\sqrt{(\sqrt{2})^2-\left(\frac{\sqrt{2}}{2}\right)^2}=\frac{\sqrt{6}}{2} \longleftarrow$$

In $\triangle$AOM, $\mathrm{AM}^2+\mathrm{OM}^2=\mathrm{OA}^2$. Also, $\mathrm{OM}=d$

$\therefore\ \mathrm{AB}=\sqrt{6} \longleftarrow$ $\boxed{\mathrm{AB}=2\mathrm{AM}}$

1. Let the points of intersection of circle $x^2+y^2=5$ and line $x+y-2=0$ be A and B. Find the length of line segment AB.



[Sol] The distance d from center $(0, 0)$ to line $x+y-2=0$ is

$$d=\frac{|0+0-2|}{\sqrt{1^2+1^2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

Also the radius is $\sqrt{5}$.

Let the midpoint of line segment AB be M.

$$\mathrm{AM}=\sqrt{(\sqrt{5})^2-(\sqrt{2})^2}=\sqrt{3}$$

$$\therefore\ \mathrm{AB}=2\sqrt{3}$$

A perpendicular dropped from the center of a circle to its chord bisects the chord.

M38b

2. Let the points of intersection of circle $(x-2)^2+(y-1)^2=4$ and line $2x+y-3=0$ be A and B. Find the length of line segment AB.

[Sol] The distance d from center $(2, 1)$ to line $2x+y-3=0$ is

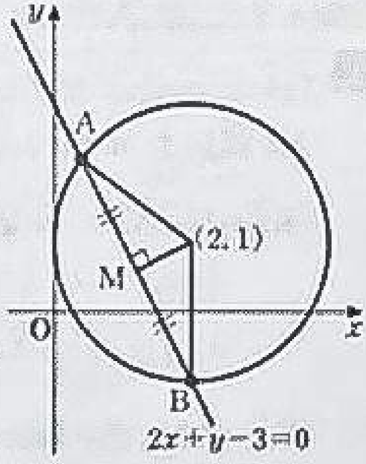
$$d=\frac{|2\cdot2+1-3|}{\sqrt{2^2+1^2}}=\frac{2}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$$

Also, the radius is 2.

Let the midpoint of line segment AB be M.

$$AM=\sqrt{2^2-\left(\frac{2\sqrt{5}}{5}\right)^2}=\frac{4\sqrt{5}}{5}$$

$$\therefore\ AB=\frac{8\sqrt{5}}{5}$$



3. Let the points of intersection of circle $x^2+y^2=8$ and line $2x+y+a=0$ be A and B, and the length of line segment AB be $2\sqrt{3}$. Find the value of a.

[Sol] The distance d from center $(0, 0)$ to line $2x+y+a=0$ is

$$d=\frac{|2\cdot0+0+a|}{\sqrt{2^2+1^2}}=\frac{|a|}{\sqrt{5}}$$

Also, the radius is $2\sqrt{2}$.

Let the midpoint of line segment AB be M.

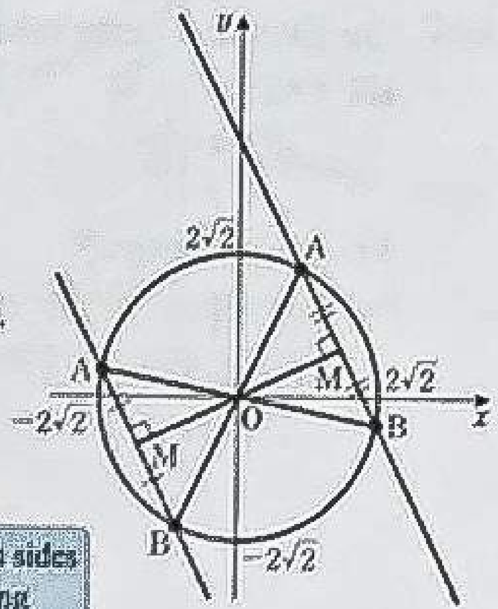
$$AM=\sqrt{(2\sqrt{2})^2-\left(\frac{a}{\sqrt{5}}\right)^2}=\sqrt{8-\frac{a^2}{5}}$$

$$\therefore\ 2\sqrt{8-\frac{a^2}{5}}=2\sqrt{3}$$

Squaring both sides and simplifying

$$a^2=25$$

$$\therefore\ a=\pm5$$



Circles 1

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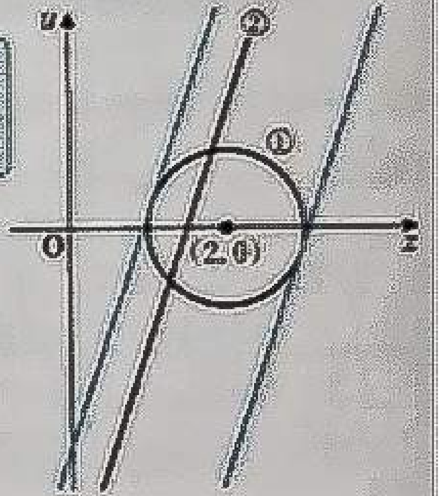
1. Find the maximum and minimum values of $y-3x+4$ for which point (x, y) moves along circle $(x-2)^2+y^2=1$ ···①.



[Sol] Let $y-3x+4=k$ ···②. ← Find the range of k for which ① and ② have common points.

From ②, $y=3x+k-4$ ···③

From ① and ③,

$$10x^2+2(3k-14)x+k^2-8k+19=0$$

$$\frac{D}{4}=(3k-14)^2-10(k^2-8k+19)$$

$$=-k^2-4k+6 \geqslant 0$$

Therefore, since $k^2+4k-6 \leqslant 0$,

$$-2-\sqrt{10} \leqslant k \leqslant -2+\sqrt{10}$$

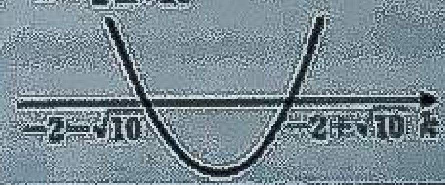
Thus,

the maximum value is $-2+\sqrt{10}$ and the minimum value is $-2-\sqrt{10}$.

From Quadratic Formula II (J I 02), the solution of $k^2+4k-6=0$ is

$$k=-2\pm\sqrt{2^2-1\cdot(-6)}$$

$$=-2\pm\sqrt{10}$$



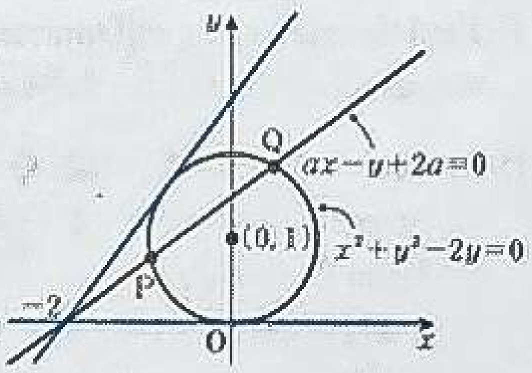
M39b

2. Given that circle $x^2+y^2-2y=0$ intersects with line $ax-y+2a=0$ at two different points P and Q, solve the following questions.

(1) Find the coordinates of the center of the circle and its radius.

[Sol] $x^2+(y-1)^2=1^2$

Therefore, **center:(0, 1), radius:1**



(2) Find the range of constant a.

[Sol] $x^2+y^2-2y=0$ ⋯①

Since $ax-y+2a=0$, $y=ax+2a$ ⋯②

From ① and ②,

$$(a^2+1)x^2+2a(2a-1)x+4a(a-1)=0$$

$$\frac{D}{4}=[a(2a-1)]^2-(a^2+1)\cdot 4a(a-1)$$

$$=-3a^2+4a>0$$ ⟵ Since intersecting at two different points, $\frac{D}{4}>0$

$$\therefore\ 3a^2-4a<0$$

$$a(3a-4)<0$$

$$\therefore\ 0<a<\frac{4}{3}$$

Alternative Solution

Let the distance from the center (0, 1) to line $ax-y+2a=0$ be d.

$$d=\frac{|a\cdot 0-1+2a|}{\sqrt{a^2+(-1)^2}}=\frac{|2a-1|}{\sqrt{a^2+1}}$$

Since the circle and the line intersect at two different points,

$$\frac{|2a-1|}{\sqrt{a^2+1}}<1$$

$$\therefore\ (2a-1)^2<a^2+1$$

$$a(3a-4)<0$$

$$\therefore\ 0<a<\frac{4}{3}$$

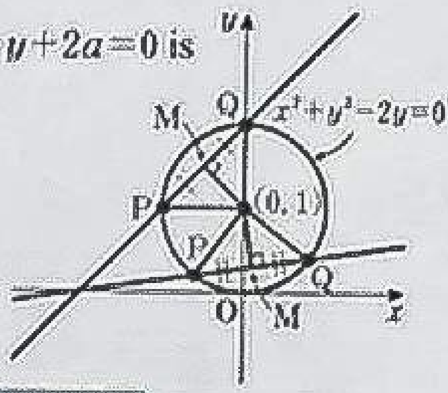
(3) Find a for which the length of PQ is $\sqrt{2}$.

[Sol] The distance d from center (0, 1) to line $ax-y+2a=0$ is

$$d=\frac{|a\cdot 0-1+2a|}{\sqrt{a^2+(-1)^2}}=\frac{|2a-1|}{\sqrt{a^2+1}}$$

Also, the radius is 1.

Let the midpoint of PQ be M.



$$PM=\sqrt{1^2-\left(\frac{2a-1}{\sqrt{a^2+1}}\right)^2}=\sqrt{\frac{-3a^2+4a}{a^2+1}}$$

$$\therefore\ \sqrt{\frac{-3a^2+4a}{a^2+1}}=\frac{\sqrt{2}}{2}$$ ⟵ Since $PM=\frac{PQ}{2}=\frac{\sqrt{2}}{2}$

$$7a^2-8a+1=0$$ ⟵ Squaring both sides and simplifying

$$(7a-1)(a-1)=0$$

$$\therefore\ a=\frac{1}{7},\ 1$$

These satisfy $0<a<\frac{4}{3}$. ⟵ From (2)

$$\therefore\ a=\frac{1}{7},\ 1$$

Circles 1

Name

Date / /

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Time : to :

1. Find the equation of the circle passing through three points A$(0, -2)$, B$(3, 7)$ and C$(7, 5)$. ➡ M34

[Sol] Let $x^2+y^2+ax+by+c=0$.

Since this circle passes through points A, B and C,

$-2b+c=-4$ …①

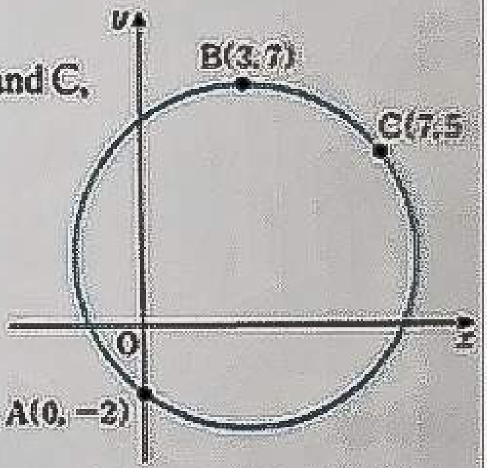
$3a+7b+c=-58$ …②

$7a+5b+c=-74$ …③

From ①~③,

$a=-6,\ b=-4,\ c=-12$

$\therefore\ x^2+y^2-6x-4y-12=0$



2. Find the equation of the circle passing through two points A$(2, -4)$ and B$(5, -3)$ with its center on line $y=x-1$. ➡ M35

[Sol] Let the center of the circle be $(a,\ a-1)$.

Let the radius of the circle be r.

$(x-a)^2+[y-(a-1)]^2=r^2$

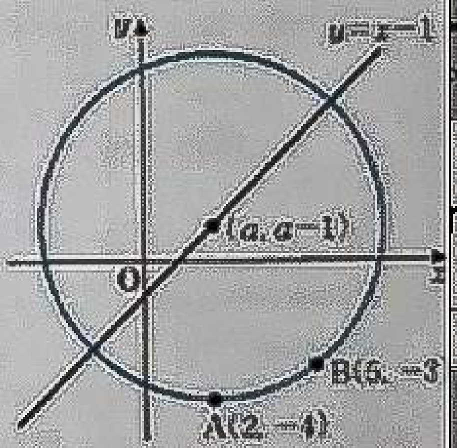
Since this circle passes through two points A and B,

$(2-a)^2+[-4-(a-1)]^2=r^2$ …①

$(5-a)^2+[-3-(a-1)]^2=r^2$ …②

From ① and ②, $a=2,\ r^2=25$

$\therefore\ (x-2)^2+(y-1)^2=25$



M40b

3. Find the range of values of k for which circle $x^2+y^2=1$ $\cdots$① intersects with line $y=2x+k$ $\cdots$② at two different points. ➡ M37

[Sol] From ① and ②,

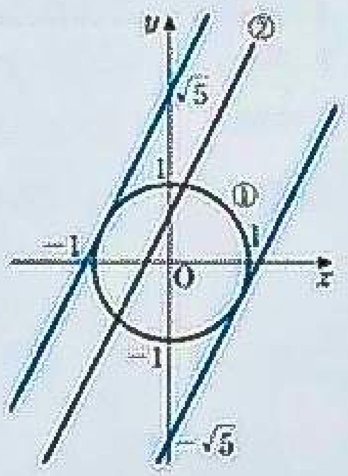
$$5x^2+4kx+k^2-1=0$$

$$\frac{D}{4}=(2k)^2-5(k^2-1)$$

$$=-k^2+5>0$$

$$\therefore\ k^2<5$$

$$\therefore\ -\sqrt{5}<k<\sqrt{5}$$



4. Let the points of intersection of circle $x^2+y^2=10$ and line $x-y-2=0$ be A and B. Find the length of line segment AB. ➡ M38

[Sol] The distance d from center $(0, 0)$ to line $x-y-2=0$ is

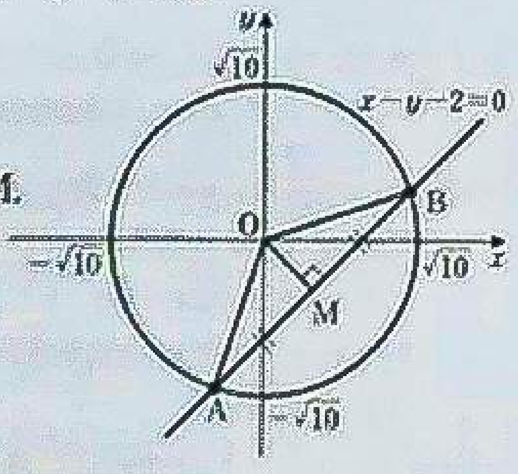
$$d=\frac{|0-0-2|}{\sqrt{1^2+(-1)^2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

Also, the radius is $\sqrt{10}$.

Let the midpoint of line segment AB be M.

$$AM=\sqrt{(\sqrt{10})^2-(\sqrt{2})^2}=2\sqrt{2}$$

$$\therefore\ AB=4\sqrt{2}$$



Alternative Solution

$x^2+y^2=10$ $\cdots$①

Since $x-y-2=0$, $y=x-2$ $\cdots$②

From ① and ②, $x^2-2x-3=0$

$$(x+1)(x-3)=0$$

$$x=-1, 3$$

Substituting into ②,

when $x=-1$, $y=-3$ and when $x=3$, $y=1$

Therefore, the points of intersection of ① and ② are $(-1, -3)$ and $(3, 1)$.

$$\therefore\ AB=\sqrt{(3+1)^2+(1+3)^2}=4\sqrt{2}$$

Circles 2

Name

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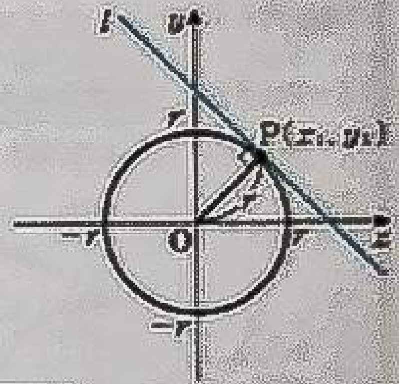
Given that line l is a tangent to circle $x^2+y^2=r^2$ at point $P(x_1, y_1)$ on the circle, show that the equation of tangent l is $x_1x+y_1y=r^2$.



[Sol] (i) When point P lies neither on the x-axis nor the y-axis, $x_1 \neq 0$ and $y_1 \neq 0$.

From the diagram, the slope of OP is $\frac{y_1}{x_1}$.

Since tangent l is perpendicular to OP,

the slope of l is $\boxed{-\frac{x_1}{y_1}}$.

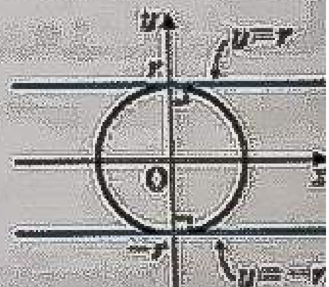
Therefore, the equation of l is $y-\boxed{y_1}=\boxed{-\frac{x_1}{y_1}}(x-\boxed{x_1})$.

Removing the denominator and simplifying,

$\boxed{x_1x+y_1y=x_1^2+y_1^2}$ …①

Since point P is a point on the circle, $x_1^2+y_1^2=\boxed{r^2}$ …②

From ① and ②, $x_1x+y_1y=r^2$ …③

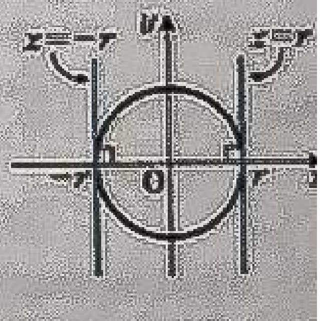


(ii) When point P lies on the y-axis,

$x_1=0$ and $y_1=\pm r$.

The equation of the tangent is either

$y=\boxed{r}$ or $y=\boxed{-r}$.



(iii) When point P lies on the x-axis,

$y_1=0$ and $x_1=\pm r$.

The equation of the tangent is either

$x=\boxed{r}$ or $x=\boxed{-r}$.

Both (ii) and (iii) satisfy ③.

From (i)~(iii), the equation of tangent l to circle $x^2+y^2=r^2$ at point $P(x_1, y_1)$ on the circle is $x_1x+y_1y=r^2$.

Answers: $-\frac{x_1}{y_1}$, y_1, $-\frac{x_1}{y_1}$, x_1, $x_1x+y_1y=x_1^2+y_1^2$, r^2, r, $-r$, r, $-r$

M41b

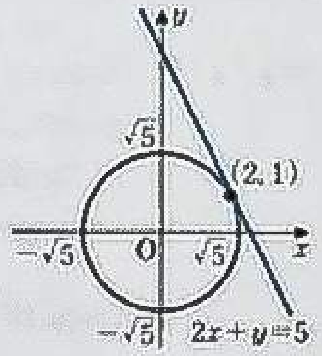
Tangent to a Circle

The equation of the tangent to circle $x^2+y^2=r^2$ at point $P(x_1, y_1)$ on the circle is

$$x_1x+y_1y=r^2$$

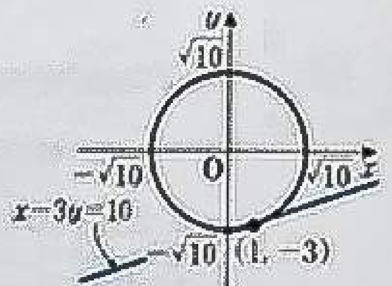
Using the formula above, solve the following questions.

(1) Find the equation of the tangent to circle $x^2+y^2=5$ at point $(2, 1)$ on the circle.



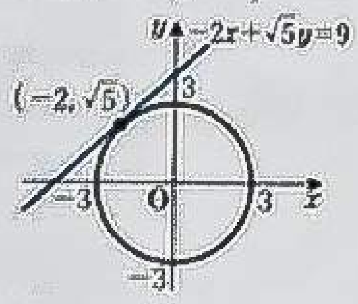
[Sol] $2x+y=5$

(2) Find the equation of the tangent to circle $x^2+y^2=10$ at point $(1, -3)$ on the circle.



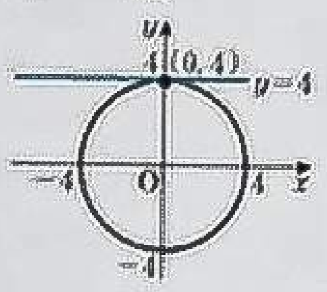
[Sol] $x-3y=10$

(3) Find the equation of the tangent to circle $x^2+y^2=9$ at point $(-2, \sqrt{5})$ on the circle.



[Sol] $-2x+\sqrt{5}y=9$

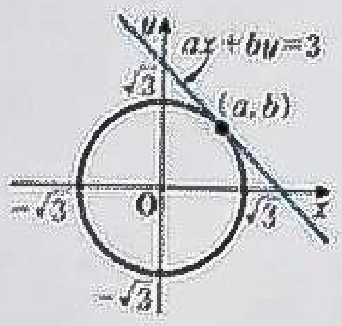
(4) Find the equation of the tangent to circle $x^2+y^2=16$ at point $(0, 4)$ on the circle.



[Sol] $0 \cdot x+4y=16$,

$\therefore\ y=4$

(5) Find the equation of the tangent to circle $x^2+y^2=3$ at point (a, b) on the circle.



[Sol] $ax+by=3$

Circles 2

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Ex. Find the equation of the line which passes through point $(3, 1)$ and is tangent to circle $x^2+y^2=5$.

[Sol] Let the tangent point be (a, b).

$a^2+b^2=5 \quad \cdots①$ ← Since (a, b) is a point on the circle

Also, the equation of the tangent is

$ax+by=5 \quad \cdots②$

Since the tangent passes through point $(3, 1)$,

$3a+b=5 \quad \cdots③$ ← Substituting $(3, 1)$ into ②

From ① and ③,

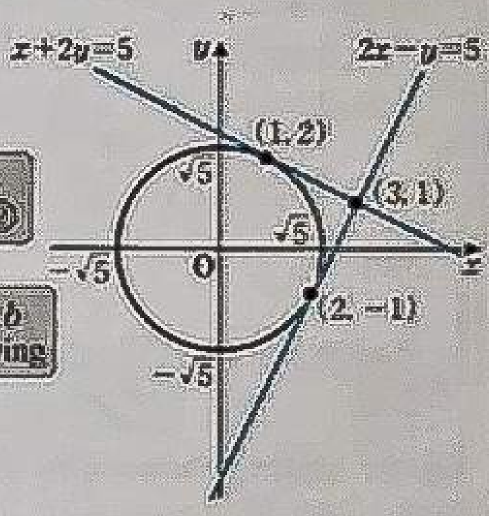
$a^2-3a+2=0$ ← Eliminating b and simplifying

$(a-1)(a-2)=0$

$\therefore\ a=1,\ 2$

From ③, when $a=1$, $b=2$ and when $a=2$, $b=-1$

Therefore, from ②, $x+2y=5$, $2x-y=5$



1. Find the equation of the line which passes through point $(7, 1)$ and is tangent to circle $x^2+y^2=25$.

[Sol] Let the tangent point be (a, b).

$a^2+b^2=25 \quad \cdots①$

Also, the equation of the tangent is

$ax+by=25 \quad \cdots②$

Since the tangent passes through point $(7, 1)$,

$7a+b=25 \quad \cdots③$

From ① and ③,

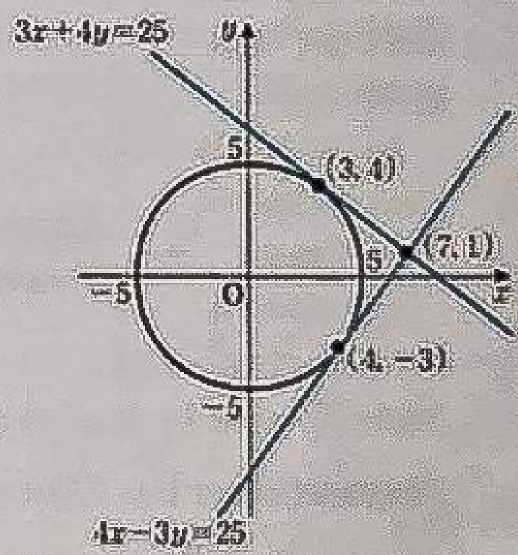
$a^2-7a+12=0$

$(a-3)(a-4)=0$

$\therefore\ a=3,\ 4$

From ③, when $a=3$, $b=4$ and when $a=4$, $b=-3$

Therefore, from ②, $3x+4y=25$, $4x-3y=25$



M42b

2. Find the equation of the line which passes through point $(-3, 1)$ and is tangent to circle $x^2+y^2=1$.

[Sol] Let the tangent point be (a, b).

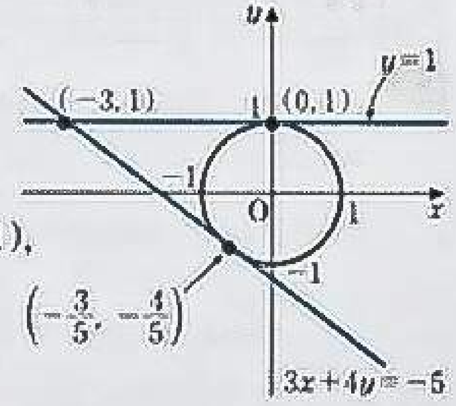
$a^2+b^2=1$ ⋯①

Also, the equation of the tangent is

$ax+by=1$ ⋯②

Since the tangent passes through point $(-3, 1)$,

$-3a+b=1$ ⋯③



From ① and ③,

$5a^2+3a=0$

$a(5a+3)=0$

$\therefore\ a=0,\ -\dfrac{3}{5}$

From ③, when $a=0$, $b=1$ and when $a=-\dfrac{3}{5}$, $b=-\dfrac{4}{5}$

Therefore, from ②, $y=1$, $3x+4y=-5$ $\left[-\dfrac{3}{5}x-\dfrac{4}{5}y=1\right]$

3. Let the tangent points of two tangents from point $(5, 10)$ to circle $x^2+y^2=25$ be A and B. Find the equation of line AB.

[Sol] Let the tangent point be (a, b).

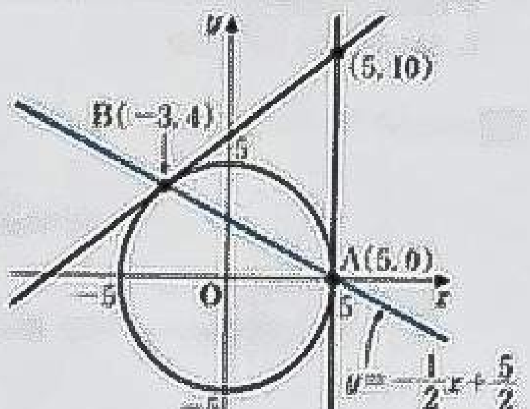
$a^2+b^2=25$ ⋯①

Also, the equation of the tangent is

$ax+by=25$

Since the tangent passes through point $(5, 10)$,

$5a+10b=25$ ⋯②



From ① and ②,

$b^2-4b=0$ ← Eliminating a and simplifying

$b(b-4)=0$

$\therefore\ b=0,\ 4$

From ②, when $b=0$, $a=5$ and when $b=4$, $a=-3$

Therefore, since line AB is $y-0=\dfrac{4-0}{-3-5}(x-5)$, ← Passing through two points $(5, 0)$ and $(-3, 4)$

$y=-\dfrac{1}{2}x+\dfrac{5}{2}$ $[x+2y=5]$

Circles 2

Name

Date / /

Time : to :

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Ex. Find the equation of the tangent which touches circle $(x-1)^2+(y-3)^2=2$ at point $(2, 4)$.

[Sol] The line $x=2$ which passes through point $(2, 4)$ and is perpendicular to the x-axis cannot be the tangent. ← From the diagram

Let the slope of the tangent be m. The equation is

$y-4=m(x-2)$

So, $mx-y-2m+4=0$ ⋯①

Let the distance from the center $(1, 3)$ to line ① be d.

$$d=\frac{|m\cdot 1-3-2m+4|}{\sqrt{m^2+(-1)^2}}=\frac{|m-1|}{\sqrt{m^2+1}} \quad \leftarrow \quad |-A|=|A|$$

Since distance d is equal to the radius,

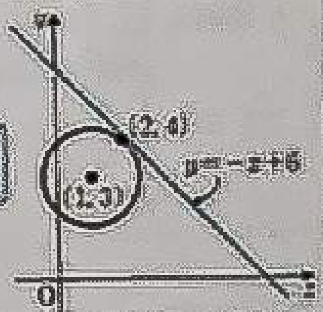
$$\frac{|m-1|}{\sqrt{m^2+1}}=\sqrt{2} \quad \cdots ②$$

Squaring both sides and simplifying

From ②, $m^2+2m+1=0$

$(m+1)^2=0$

$\therefore m=-1 \quad \therefore y=-x+6$



1. Find the equation of the tangent which touches circle $(x+3)^2+(y-2)^2=34$ at point $(2, 5)$.

[Sol] The line $x=2$ which passes through point $(2, 5)$ and is perpendicular to the x-axis cannot be the tangent.

Let the slope of the tangent be m. The equation is

$y-5=m(x-2)$

So, $mx-y-2m+5=0$ ⋯①

Let the distance from the center $(-3, 2)$ to line ① be d.

$$d=\frac{|m\cdot(-3)-2-2m+5|}{\sqrt{m^2+(-1)^2}}=\frac{|5m-3|}{\sqrt{m^2+1}}$$

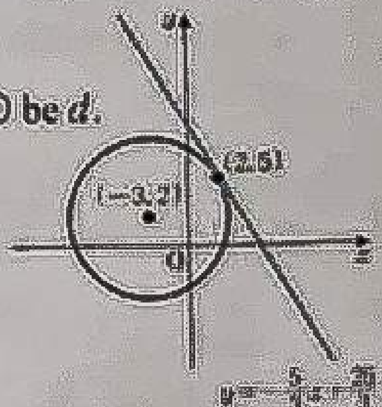
Since distance d is equal to the radius,

$$\frac{|5m-3|}{\sqrt{m^2+1}}=\sqrt{34} \quad \cdots ②$$

From ②, $9m^2+30m+25=0$

$(3m+5)^2=0$

$\therefore m=-\frac{5}{3} \quad \therefore y=-\frac{5}{3}x+\frac{25}{3}$ $\boxed{5x+3y-25=0}$



M43b

2. Find the equation of the line which passes through point $(3, 4)$ and is tangent to circle $(x+1)^2+(y-2)^2=10$.

[Sol] The line $x=3$ which passes through point $(3, 4)$ and is perpendicular to the x-axis cannot be the tangent.

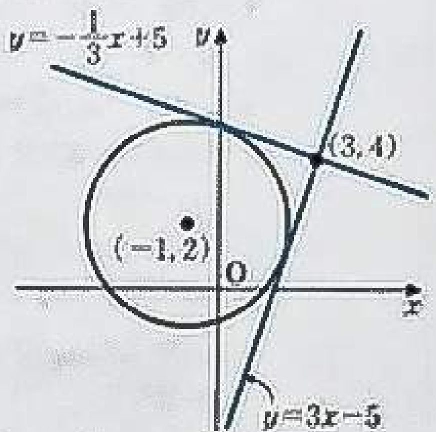
Let the slope of the tangent be m. The equation is

$y-4=m(x-3)$

So, $mx-y-3m+4=0$ ⋯①

Let the distance from the center $(-1, 2)$ to line ① be d.

$$d=\frac{|m\cdot(-1)-2-3m+4|}{\sqrt{m^2+(-1)^2}}=\frac{|4m-2|}{\sqrt{m^2+1}}$$



Since distance d is equal to the radius,

$$\frac{|4m-2|}{\sqrt{m^2+1}}=\sqrt{10} \quad \cdots②$$

From ②, $3m^2-8m-3=0$

$(3m+1)(m-3)=0$

$\therefore\ m=-\frac{1}{3},\ 3$

$\therefore\ y=-\frac{1}{3}x+5,\ y=3x-5\quad [x+3y-15=0,\ 3x-y-5=0]$

3. Find the equation of the line which passes through point $(3, 3)$ and is tangent to circle $(x-2)^2+(y-1)^2=1$.

[Sol] The line $x=3$ which passes through point $(3, 3)$ and is perpendicular to the x-axis can be the tangent. ⟵ From the diagram

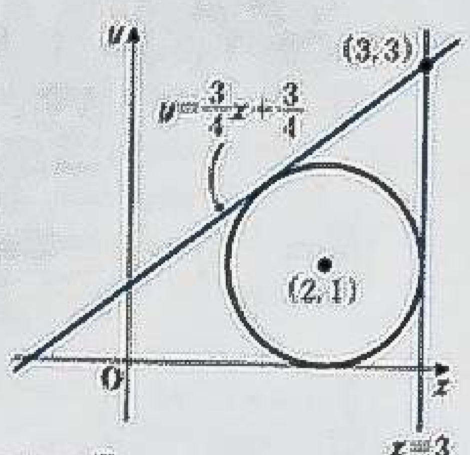
Let the slope of the tangent be m. The equation is

$y-3=m(x-3)$

So, $mx-y-3m+3=0$ ⋯①

Let the distance from the center $(2, 1)$ to line ① be d.

$$d=\frac{|m\cdot2-1-3m+3|}{\sqrt{m^2+(-1)^2}}=\frac{|m-2|}{\sqrt{m^2+1}}$$



Since distance d is equal to the radius,

$$\frac{|m-2|}{\sqrt{m^2+1}}=1 \quad \cdots②$$

From ②, $m=\frac{3}{4}$

$\therefore\ x=3,\ y=\frac{3}{4}x+\frac{3}{4}\quad [x=3,\ 3x-4y+3=0]$

Circles 2

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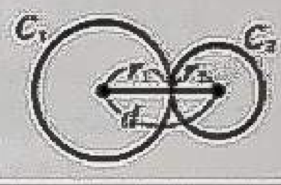
Internally/Externally Tangent Circles

Let the radii of two circles C_1 and C_2 be r_1 and $r_2 (r_1 > r_2)$ respectively, and the distance between the centers of the circles be d.

[1] Internally tangent $d = r_1 - r_2$

[2] Externally tangent $d = r_1 + r_2$



Ex. Find the equation of the circle with center at point $(1, \sqrt{3})$ which is internally tangent to circle $x^2+y^2=25$ ···①.

[Sol] Circle ① has its center at point (0, 0) and radius 5. The distance d between the centers of the circles is

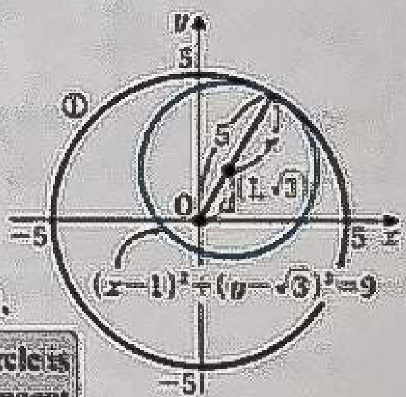
$d=\sqrt{1^2+(\sqrt{3})^2}=2$

Let the radius of the circle be r. Since this circle is internally tangent to circle ①,

$2=5-r$ ⟵ When the circle is internally tangent to circle ①, $5>r$

$\therefore\ r=3$

$\therefore\ (x-1)^2+(y-\sqrt{3})^2=9$



1. Find the equation of the circle with center at point (1, 2) which is internally tangent to circle $x^2+y^2=20$ ···①.

[Sol] Circle ① has its center at point (0, 0) and radius $2\sqrt{5}$.
The distance d between the centers of the circles is

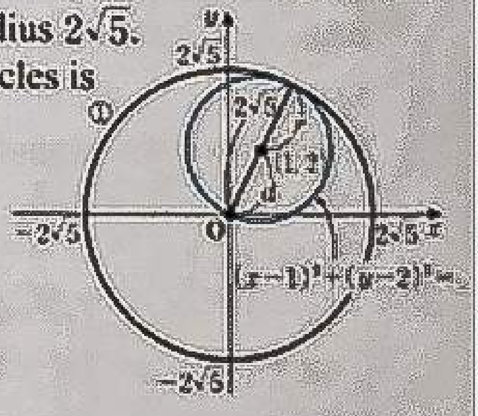
$d=\sqrt{1^2+2^2}=\sqrt{5}$

Let the radius of the circle be r. Since this circle is internally tangent to circle ①,

$\sqrt{5}=2\sqrt{5}-r$

$\therefore\ r=\sqrt{5}$

$\therefore\ (x-1)^2+(y-2)^2=5$



As shown in [1] and [2] above, when two circles have only one common point, it is said th the two circles are ***tangent***. [1] : ***internally tangent*** [2] : ***externally tangent***

M44b

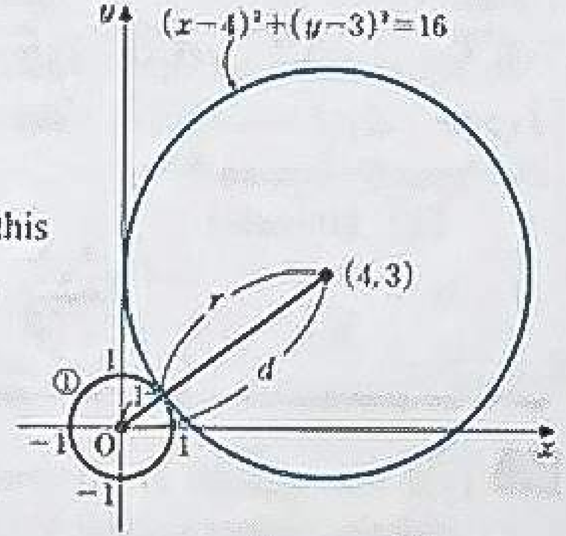
2. Find the equation of the circle with center at point $(4, 3)$ which is externally tangent to circle $x^2+y^2=1$ ⋯①.

[Sol] Circle ① has its center at point $(0, 0)$ and radius 1. The distance d between the centers of the circles is

$$d=\sqrt{4^2+3^2}=5$$

Let the radius of the circle be r. Since this circle is externally tangent to circle ①,

$$5=r+1$$
$$\therefore\ r=4$$
$$\therefore\ (x-4)^2+(y-3)^2=16$$



3. Find the equation of the circle with center at point $(-1, 7)$ which is tangent to circle $x^2+y^2-8x+10y+16=0$ ⋯①.

[Sol] Rearranging circle ①,

$$(x-4)^2+(y+5)^2=25$$

Circle ① has its center at point $(4, -5)$ and radius 5.

The distance d between the centers of the circles is

$$d=\sqrt{(4+1)^2+(-5-7)^2}=13$$

Let the radius of the circle be r.

When externally tangent,

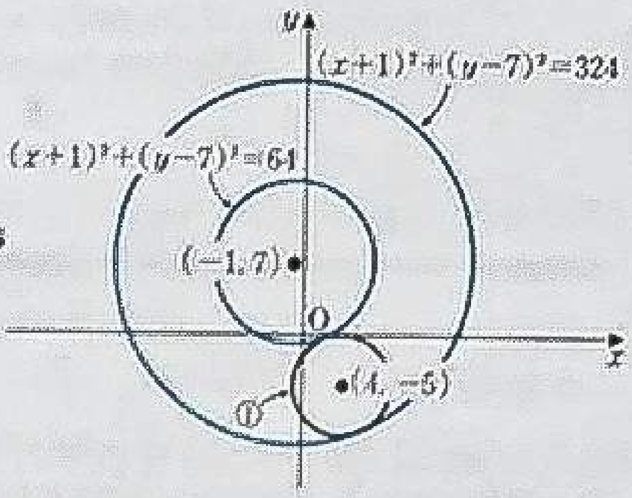
$13=r+5 \quad \therefore\ r=8$

When internally tangent,

$13=r-5 \quad \therefore\ r=18$ ← When circle ① is internally tangent to the circle, $r>5$

Therefore,

$$(x+1)^2+(y-7)^2=64$$
$$(x+1)^2+(y-7)^2=324$$



Circles 2

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Ex. Find the value of k when two circles $x^2+y^2=5$ $\cdots$① and $x^2+y^2-8x-4y+17-k=0$ $\cdots$② are externally tangent.

[Sol] From ①, circle ① has its center at point $(0, 0)$ and radius $\sqrt{5}$.

From ②, $(x-4)^2+(y-2)^2=k+3$

$\therefore k>-3$ $\cdots$③ ← The condition for ② to represent a circle is $k+3>0$.

Circle ② has its center at point $(4, 2)$ and radius $\sqrt{k+3}$.

The distance d between the centers of the circles is

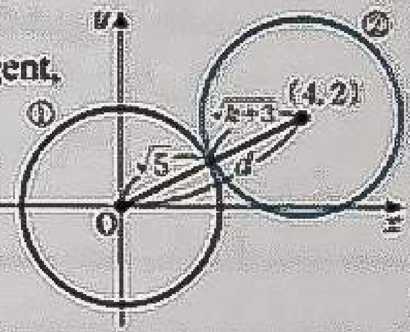
$d=\sqrt{4^2+2^2}=2\sqrt{5}$

Since circles ① and ② are externally tangent,

$2\sqrt{5}=\sqrt{5}+\sqrt{k+3}$ $\quad\therefore k=2$ $\cdots$④

④ satisfies ③.

$\therefore k=2$



1. Find the value of k when two circles $(x+1)^2+(y-1)^2=4$ $\cdots$① and $x^2+y^2-6x-8y+k=0$ $\cdots$② are externally tangent.

[Sol] From ①, circle ① has its center at point $(-1, 1)$ and radius 2.

From ②, $(x-3)^2+(y-4)^2=25-k$

$\therefore k<25$ $\cdots$③

Circle ② has its center at point $(3, 4)$ and radius $\sqrt{25-k}$.

The distance d between the centers of the circles is

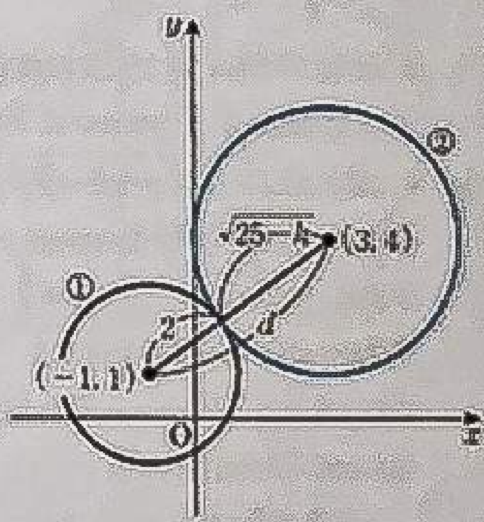
$d=\sqrt{(3+1)^2+(4-1)^2}=5$

Since circles ① and ② are externally tangent,

$5=2+\sqrt{25-k}$ $\quad\therefore k=16$ $\cdots$④

④ satisfies ③.

$\therefore k=16$



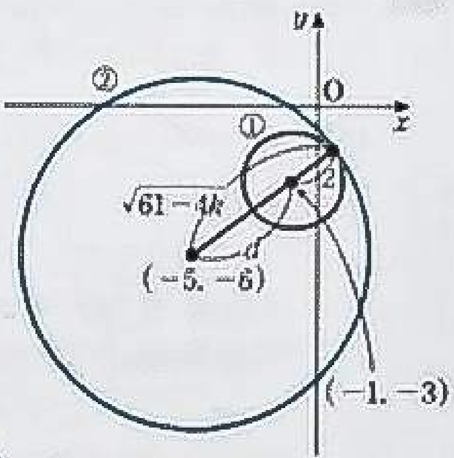
M45b

2. Find the value of k when two circles $(x+1)^2+(y+3)^2=4$ …① and $x^2+y^2+10x+12y+4k=0$ …② are internally tangent.

[Sol] From ①, circle ① has its center at point $(-1, -3)$ and radius 2.

From ②, $(x+5)^2+(y+6)^2=61-4k$

$\therefore\ k<\dfrac{61}{4}$ …③



Circle ② has its center at point $(-5, -6)$ and radius $\sqrt{61-4k}$.

The distance d between the centers of the circles is

$d=\sqrt{(-5+1)^2+(-6+3)^2}=5$

Since circles ① and ② are internally tangent,

$5=\sqrt{61-4k}-2$ ← From diagram, since circle ① is internally tangent to circle ②, $\sqrt{61-4k}>2$

$\therefore\ k=3$ …④

④ satisfies ③.

$\therefore\ k=3$

3. Find the value of k when two circles $(x+1)^2+(y-2)^2=4$ …① and $x^2+y^2-4x-12y+32-k=0$ …② are tangent to each other.

[Sol] From ①, circle ① has its center at point $(-1, 2)$ and radius 2.

From ②, $(x-2)^2+(y-6)^2=k+8$

$\therefore\ k>-8$ …③

Circle ② has its center at point $(2, 6)$ and radius $\sqrt{k+8}$.

The distance d between the centers of the circles is

$d=\sqrt{(2+1)^2+(6-2)^2}=5$

When externally tangent,

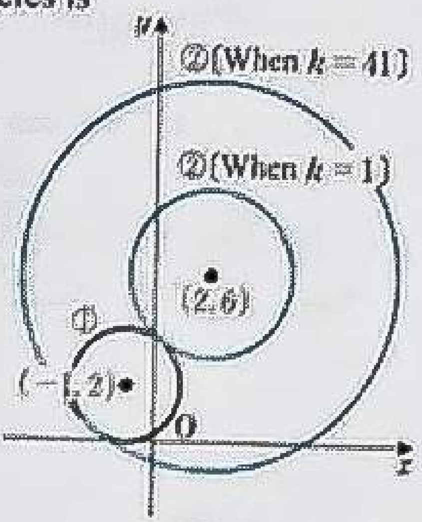
$5=2+\sqrt{k+8}$ $\quad\therefore\ k=1$ …④

When internally tangent,

$5=\sqrt{k+8}-2$ $\quad\therefore\ k=41$ …⑤

④ and ⑤ satisfy ③.

$\therefore\ k=1,\ 41$



Circles 2

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Ex. Find the coordinates of the common points of two circles
$x^2+y^2-10=0$ ···① and $x^2+y^2-8x+4y+10=0$ ···②.

[Sol] From ① and ②,

$8x-4y-20=0$ ⟵ From ①−②

$\therefore\ y=2x-5$ ···③

From ① and ③,

$x^2-4x+3=0$ ⟵ Substituting ③ into ① and simplifying

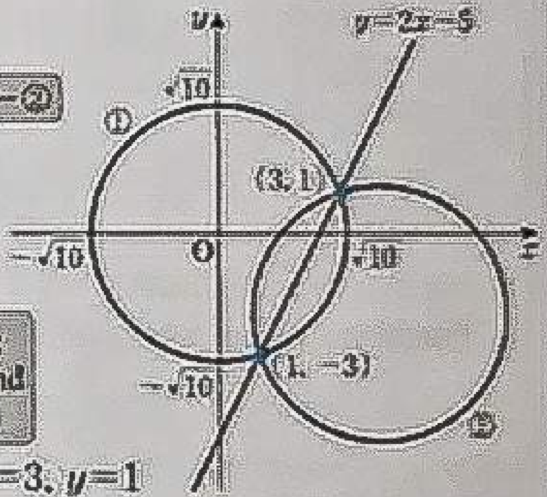
$(x-1)(x-3)=0$

$\therefore\ x=1,\ 3$

From ③,

when $x=1$, $y=-3$ and when $x=3$, $y=1$

Therefore, the common points are $(1,\ -3)$, $(3,\ 1)$.



1. Find the coordinates of the common points of two circles $x^2+y^2-5=0$ ···① and $x^2+y^2-6x-2y+5=0$ ···②.

[Sol] From ① and ②,

$6x+2y-10=0$

$\therefore\ y=-3x+5$ ···③

From ① and ③,

$x^2-3x+2=0$

$(x-1)(x-2)=0$

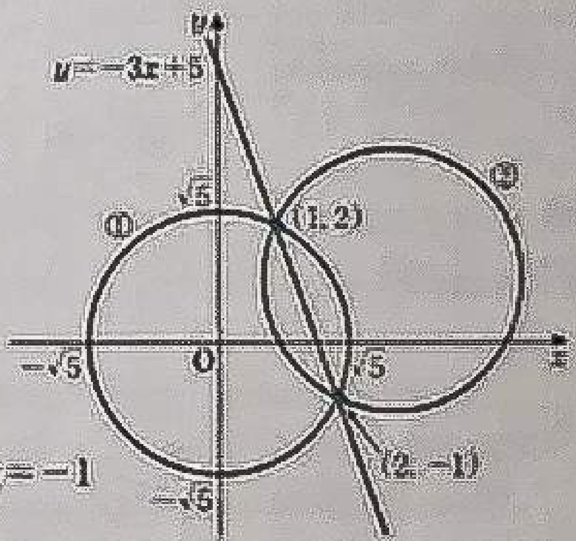
$\therefore\ x=1,\ 2$

From ③,

when $x=1$, $y=2$ and when $x=2$, $y=-1$

Therefore, the common points are

$(1,\ 2)$, $(2,\ -1)$.



M46b

2. Find the coordinates of the common points of two circles
$x^2+y^2-2x-4y+1=0$ …① and $x^2+y^2-4x-6y+11=0$ …②.

[Sol] From ① and ②,

$2x+2y-10=0$

$\therefore\ y=-x+5$ …③

From ① and ③,

$x^2-4x+3=0$

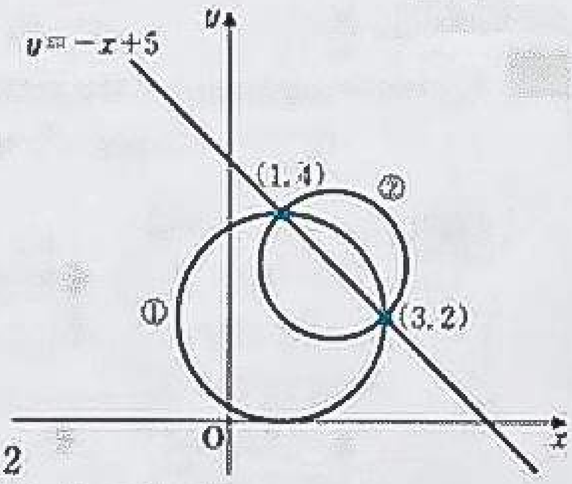
$(x-1)(x-3)=0$

$\therefore\ x=1,\ 3$

From ③,

when $x=1$, $y=4$ and when $x=3$, $y=2$

Therefore, the common points are **(1, 4), (3, 2)**.



3. Let the points of intersection of two circles $x^2+y^2-2y=0$ …① and $x^2+y^2-2x-4=0$ …② be points P and Q. Find the equation of the circle which passes through points P, Q and R(2, 0).

[Sol] From ① and ②,

$2x-2y+4=0$

$\therefore\ y=x+2$ …③

From ① and ③,

$x^2+x=0$

$x(x+1)=0$

$\therefore\ x=0,\ -1$

From ③,

when $x=0$, $y=2$ and when $x=-1$, $y=1$

Therefore, the points of intersection are $(0, 2)$, $(-1, 1)$.

Let the equation of the circle be $x^2+y^2+ax+by+c=0$. ← M34

Since this circle passes through points P, Q and R,

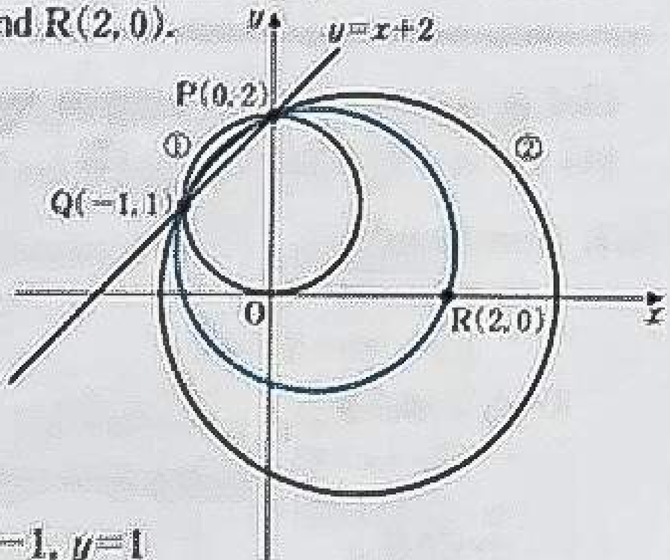
$2b+c=-4$ …④

$-a+b+c=-2$ …⑤

$2a+c=-4$ …⑥

From ④~⑥, $a=-1$, $b=-1$, $c=-2$

$\therefore\ x^2+y^2-x-y-2=0$



Circles 2

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Let A be the point of intersection of two lines:

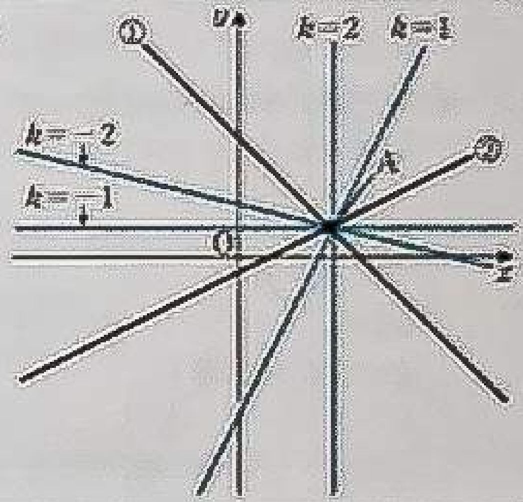
$x+y-4=0 \quad \cdots ①$

$x-2y-1=0 \quad \cdots ②$

Since point A lies on lines ① and ②,

$k(x+y-4)+(x-2y-1)=0$

represents a line passing through the point of intersection A regardless of the value of constant k.



Ex. Find the equation of the line which passes through point $(3, 2)$ and the point of intersection of two lines $2x-y-3=0$ and $x+2y-4=0$.

[Sol] Let k be a constant.

$k(2x-y-3)+(x+2y-4)=0$

Since the line passes through point $(3, 2)$,

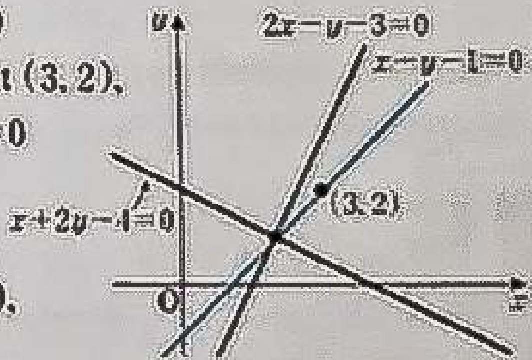
$k(2\cdot3-2-3)+(3+2\cdot2-4)=0$

$\therefore\ k=-3$

Therefore, since

$-3(2x-y-3)+(x+2y-4)=0$,

$x-y-1=0$



1. Find the equation of the line which passes through point $(4, 3)$ and the point of intersection of two lines $x+y-4=0$ and $x-2y-1=0$.

[Sol] Let k be a constant.

$k(x+y-4)+(x-2y-1)=0$

Since the line passes through point $(4, 3)$,

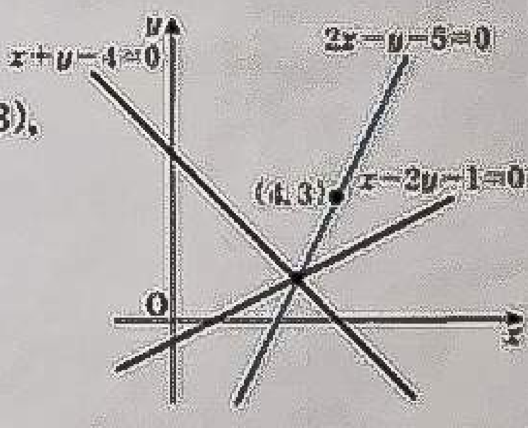
$k(4+3-4)+(4-2\cdot3-1)=0$

$\therefore\ k=1$

Therefore, since

$1\cdot(x+y-4)+(x-2y-1)=0$,

$2x-y-5=0$



M47b

2. Find the equation of the line which passes through point (6, 3) and the point of intersection of two lines $x-2y+1=0$ and $3x+4y-17=0$.

[Sol] Let k be a constant.

$$k(x-2y+1)+(3x+4y-17)=0$$

Since the line passes through point (6, 3),

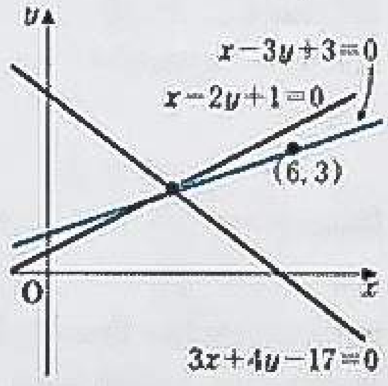
$$k(6-2\cdot3+1)+(3\cdot6+4\cdot3-17)=0$$

$$\therefore\ k=-13$$

Therefore, since

$$-13(x-2y+1)+(3x+4y-17)=0,$$

$$x-3y+3=0$$



3. Find the equation of the line which passes through point (2, −1) and the point of intersection of two lines $x+7y+1=0$ and $2x-3y-1=0$.

[Sol] Let k be a constant.

$$k(x+7y+1)+(2x-3y-1)=0$$

Since the line passes through point (2, −1),

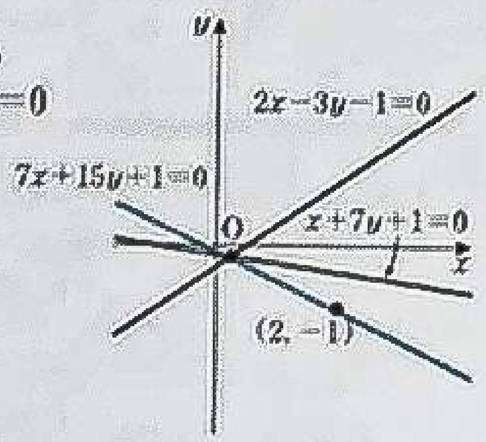
$$k[2+7\cdot(-1)+1]+[2\cdot2-3\cdot(-1)-1]=0$$

$$\therefore\ k=\frac{3}{2}$$

Therefore, since

$$\frac{3}{2}(x+7y+1)+(2x-3y-1)=0,$$

$$7x+15y+1=0$$



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Let A and B be the points of intersection of two circles:

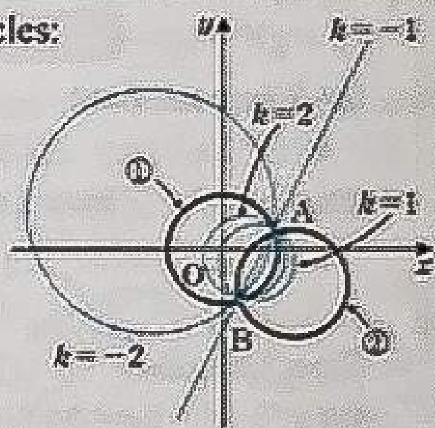
$x^2+y^2-10=0$ ···①

$x^2+y^2-8x+4y+10=0$ ···②

Since points A and B lie on circles ① and ②,

$k(x^2+y^2-10)+(x^2+y^2-8x+4y+10)=0$

represents a figure which passes through the points of intersection A and B regardless of the value of constant k.



※ When $k=-1$, the equation represents a line which passes through the points of intersection A and B of two circles.

Ex. Find the equation of the circle which passes through point (3, 2) and the two points of intersection of two circles $x^2+y^2-10=0$ and $x^2+y^2-4x-2y=0$.

[Sol] Let k be a constant.

$k(x^2+y^2-10)+(x^2+y^2-4x-2y)=0$

Since the circle passes through point (3, 2),

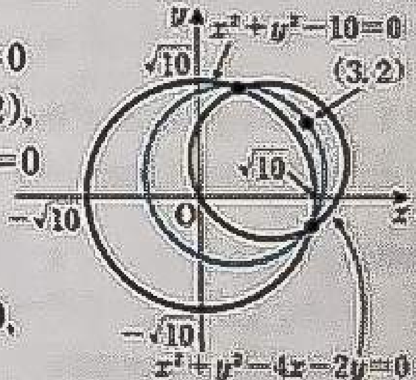
$k(3^2+2^2-10)+(3^2+2^2-4\cdot3-2\cdot2)=0$

$\therefore\ k=1$

Therefore, since

$1\cdot(x^2+y^2-10)+(x^2+y^2-4x-2y)=0,$

$x^2+y^2-2x-y-5=0$



1. Find the equation of the circle which passes through point (1, 0) and the tw points of intersection of two circles $x^2+y^2-2=0$ and $x^2+y^2-6x-2y+6=($

[Sol] Let k be a constant.

$k(x^2+y^2-2)+(x^2+y^2-6x-2y+6)=0$

Since the circle passes through point (1, 0),

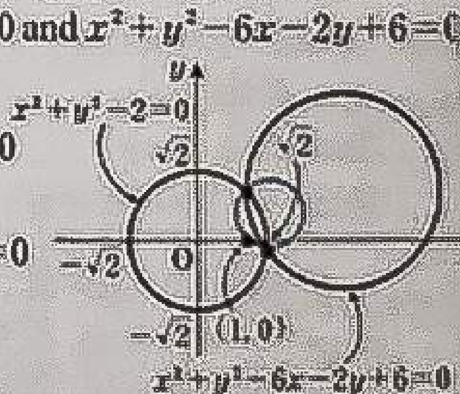
$k(1^2+0^2-2)+(1^2+0^2-6\cdot1-2\cdot0+6)=0$

$\therefore\ k=1$

Therefore, since

$1\cdot(x^2+y^2-2)+(x^2+y^2-6x-2y+6)=0,$

$x^2+y^2-3x-y+2=0$



M48b

2. Find the radius of the circle which passes through point $(3, 1)$ and the two points of intersection of two circles $(x-2)^2+(y-1)^2=5$ and $(x-1)^2+(y-2)^2=1$.

[Sol] $(x-2)^2+(y-1)^2=5$, i.e. $x^2+y^2-4x-2y=0$

$(x-1)^2+(y-2)^2=1$, i.e. $x^2+y^2-2x-4y+4=0$

Let k be a constant.

$$k(x^2+y^2-4x-2y)+(x^2+y^2-2x-4y+4)=0$$

Since the circle passes through point $(3, 1)$,

$$k(3^2+1^2-4\cdot3-2\cdot1)+(3^2+1^2-2\cdot3-4\cdot1+4)=0$$

$$\therefore k=1$$

Therefore, since

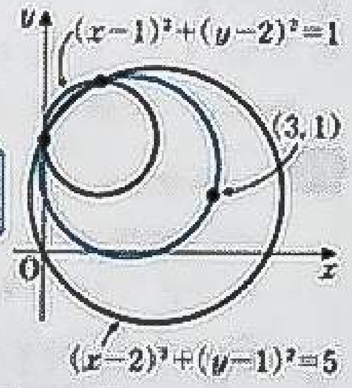
$1\cdot(x^2+y^2-4x-2y)+(x^2+y^2-2x-4y+4)=0$,

$$x^2+y^2-3x-3y+2=0$$

$$\left(x-\frac{3}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=\frac{5}{2}$$

Thus, the radius is $\dfrac{\sqrt{10}}{2}$.

From $\dfrac{5}{2}=\left(\dfrac{\sqrt{10}}{2}\right)^2$



3. Find the equation of the circle which passes through the origin and the point of intersection of line $x+y=3$ and circle $(x-1)^2+(y-3)^2=1$.

[Sol] $x+y=3$, i.e. $x+y-3=0$

$(x-1)^2+(y-3)^2=1$, i.e. $x^2+y^2-2x-6y+9=0$

Let k be a constant.

$$k(x+y-3)+(x^2+y^2-2x-6y+9)=0$$

Since the circle passes through the origin,

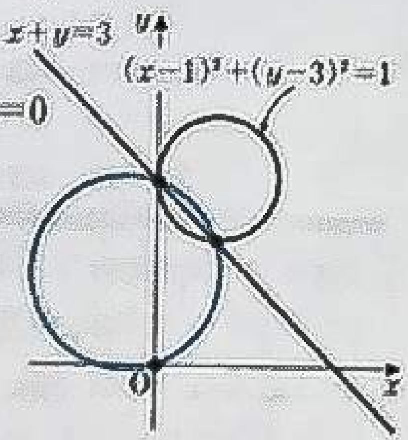
$$k(0+0-3)+(0^2+0^2-2\cdot0-6\cdot0+9)=0$$

$$\therefore k=3$$

Therefore, since

$3(x+y-3)+(x^2+y^2-2x-6y+9)=0$,

$$x^2+y^2+x-3y=0 \quad \left[\left(x+\frac{1}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=\frac{5}{2}\right]$$



Circles 2

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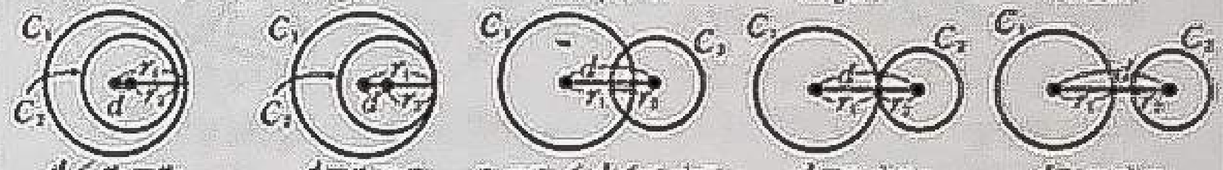
Positional Relationship between Two Circles

Let the radii of two circles C_1 and C_2 be r_1 and $r_2 (r_1 > r_2)$ respectively and the distance between the centers of the circles be d.

[1] Completely inside [2] Internally tangent [3] Intersect at two points [4] Externally tangent [5] Completely outside



$d < r_1 - r_2$ $d = r_1 - r_2$ $r_1 - r_2 < d < r_1 + r_2$ $d = r_1 + r_2$ $d > r_1 + r_2$

※ [3]~[5] are also true when $r_1 = r_2$. If it is not possible to define which is greater between the values of r_1 and r_2, use $|r_1 - r_2|$ instead of $r_1 - r_2$ in [1]~[3].

1. Find the range of values of a for which two circles $x^2+y^2=1$ and $(x-a)^2+(y-1)^2=1$ intersect at two different points.

[Sol] Circle $x^2+y^2=1$ has its center at point $(0, 0)$ and radius 1. Also, circle $(x-a)^2+(y-1)^2=1$ has its center at point $(a, 1)$ and radius 1. The distance d between the centers of the two circles is $d=\sqrt{a^2+1}$.

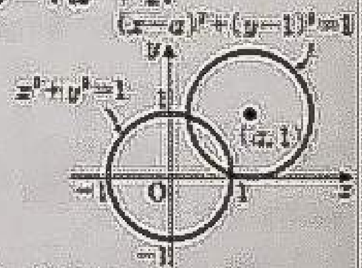
$\therefore\ 1-1 < \sqrt{a^2+1} < 1+1$

$0 < \sqrt{a^2+1} < 2$

$0 < \sqrt{a^2+1}$, i.e. it is true with any real numbers.

$\sqrt{a^2+1} < 2$, i.e. $-\sqrt{3} < a < \sqrt{3}$ ⟵ $a^2+1<4$, i.e. $a^2<3$

$\therefore\ -\sqrt{3} < a < \sqrt{3}$



2. Find the range of values of a for which two circles $x^2+y^2=1$ and $(x-a)^2+y^2=\frac{a^2}{4} (a > 0)$ intersect at two different points.

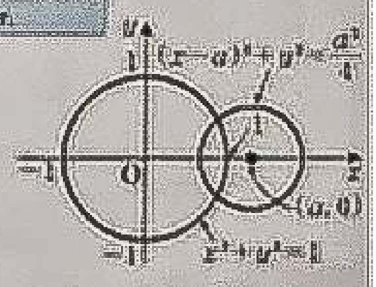
[Sol] Circle $x^2+y^2=1$ has its center at point $(0, 0)$ and radius 1. Also, circle $(x-a)^2+y^2=\frac{a^2}{4}$ has its center at point $(a, 0)$ and radius $\frac{a}{2}$. The distance d between the centers of the two circles is $d=a$.

$\therefore\ \left|1-\frac{a}{2}\right| < a < 1+\frac{a}{2}$ ⟵ Use the absolute value, since it is not possible to define which value of the two radii is greater.

$\left|1-\frac{a}{2}\right| < a$, i.e. $a > \frac{2}{3}$ ⟵ $-a < 1-\frac{a}{2} < a$, i.e. $-\frac{1}{2}a < 1 < \frac{3}{2}a$

$a < 1+\frac{a}{2}$, i.e. $a < 2$

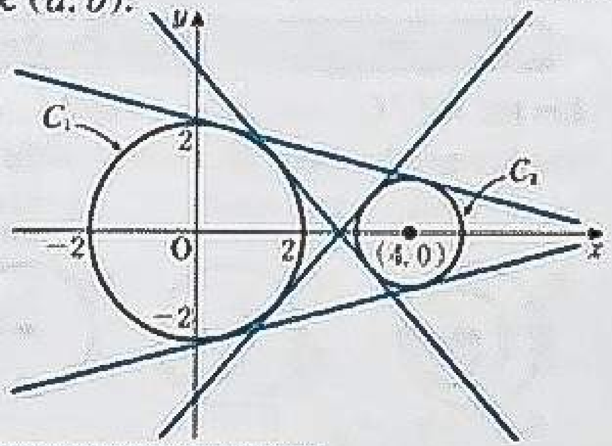
$\therefore\ \frac{2}{3} < a < 2$



M49b

3. Given four lines which are tangent to both circles $C_1: x^2+y^2=4$ and $C_2: (x-4)^2+y^2=1$, find the equations of those four lines.

[Sol] Let the tangent point on circle C_1 be (a, b).



$a^2+b^2=4$ ⋯①

Also, the equation of the tangent is

$ax+by=4$

So, $ax+by-4=0$ ⋯②

When line ② is also tangent to circle C_2, let the distance between the center of the circle $(4, 0)$ and line ② be d.

$d=\dfrac{|4a-4|}{\sqrt{a^2+b^2}}$ ⟵ Since $\dfrac{|a\cdot 4+b\cdot 0-4|}{\sqrt{a^2+b^2}}=\dfrac{|4a-4|}{\sqrt{a^2+b^2}}$

Since distance d is equal to radius 1,

$\dfrac{|4a-4|}{\sqrt{a^2+b^2}}=1$ ⋯③

From ① and ③,

$(4a-4)^2=4$ ⟵ From ③, $|4a-4|=\sqrt{a^2+b^2}$ Squaring both sides, $(4a-4)^2=a^2+b^2$

$\therefore\ 4a^2-8a+3=0$

$(2a-1)(2a-3)=0$

$\therefore\ a=\dfrac{1}{2}, \dfrac{3}{2}$

From ①, when $a=\dfrac{1}{2}$, $b=\pm\dfrac{\sqrt{15}}{2}$ and when $a=\dfrac{3}{2}$, $b=\pm\dfrac{\sqrt{7}}{2}$

Therefore, from ②, $\dfrac{1}{2}x\pm\dfrac{\sqrt{15}}{2}y-4=0$, $\dfrac{3}{2}x\pm\dfrac{\sqrt{7}}{2}y-4=0$

$\therefore\ x\pm\sqrt{15}y-8=0$, $3x\pm\sqrt{7}y-8=0$

Circles 2

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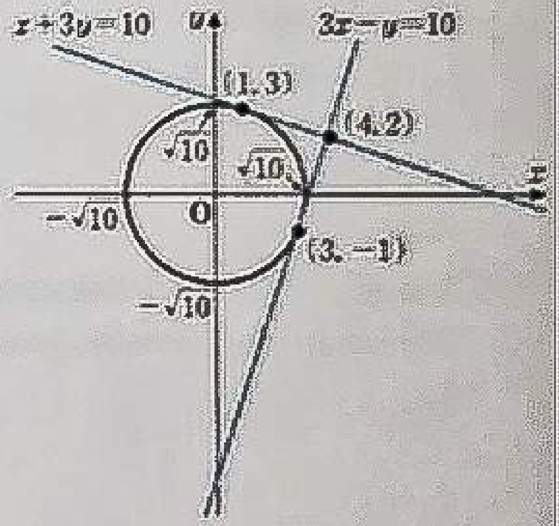
1. Find the equation of the line which passes through point $(4, 2)$ and is tangent to circle $x^2+y^2=10$. ➡ M42



[Sol] Let the tangent point be (a, b).

$a^2+b^2=10$ ···①

Also, the equation of the tangent is

$ax+by=10$ ···②

Since the tangent passes through point $(4, 2)$,

$4a+2b=10$ ···③

From ① and ③,

$a^2-4a+3=0$

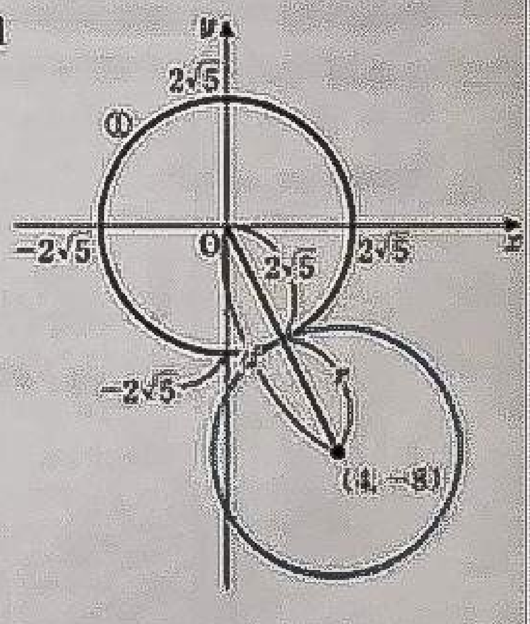
$(a-1)(a-3)=0$

$\therefore\ a=1,\ 3$

From ③, when $a=1$, $b=3$ and when $a=3$, $b=-1$

Therefore, from ②, $x+3y=10$, $3x-y=10$

2. Find the equation of the circle with center at point $(4, -8)$ which is externally tangent to circle $x^2+y^2=20$ ···①. ➡ M44



[Sol] Circle ① has its center at point $(0, 0)$ and radius $2\sqrt{5}$. The distance d between the centers of the circles is

$d=\sqrt{4^2+(-8)^2}=4\sqrt{5}$

Let the radius of the circle be r.

Since this circle is externally tangent to circle ①,

$4\sqrt{5}=r+2\sqrt{5}$

$\therefore\ r=2\sqrt{5}$

$\therefore\ (x-4)^2+(y+8)^2=20$

M50b

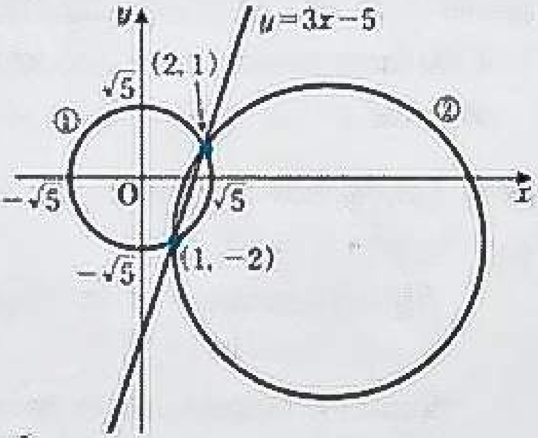
3. Find the coordinates of the common points of two circles $x^2+y^2-5=0$ $\cdots$①
and $x^2+y^2-12x+4y+15=0$ $\cdots$②.

➡ M46

[Sol] From ① and ②,

$$12x-4y-20=0$$

$$\therefore\ y=3x-5 \quad \cdots ③$$



From ① and ③,

$$x^2-3x+2=0$$

$$(x-1)(x-2)=0$$

$$\therefore\ x=1,\ 2$$

From ③,

when $x=1$, $y=-2$ and when $x=2$, $y=1$

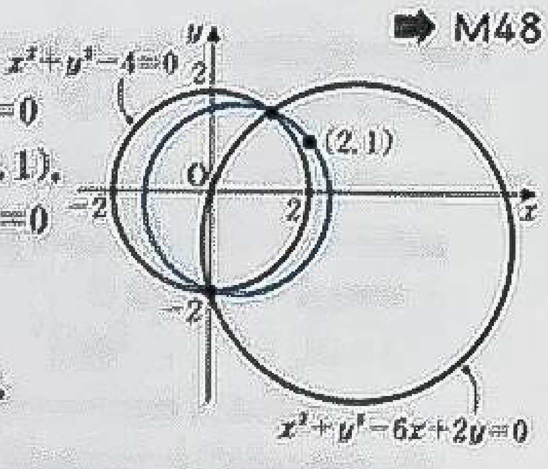
Therefore, the common points are $(1,\ -2)$, $(2,\ 1)$.

4. Find the equation of the circle which passes through point $(2,\ 1)$ and the two points of intersection of two circles $x^2+y^2-4=0$ and $x^2+y^2-6x+2y=0$.

➡ M48

[Sol] Let k be a constant.

$$k(x^2+y^2-4)+(x^2+y^2-6x+2y)=0$$



Since the circle passes through point $(2,\ 1)$,

$$k(2^2+1^2-4)+(2^2+1^2-6\cdot 2+2\cdot 1)=0$$

$$\therefore\ k=5$$

Therefore, since

$$5(x^2+y^2-4)+(x^2+y^2-6x+2y)=0,$$

$$x^2+y^2-x+\frac{1}{3}y-\frac{10}{3}=0$$

$$\boxed{3x^2+3y^2-3x+y-10=0}$$

Loci 1

Name

Date / /

100%	~90%	~80%	~70%	69%~
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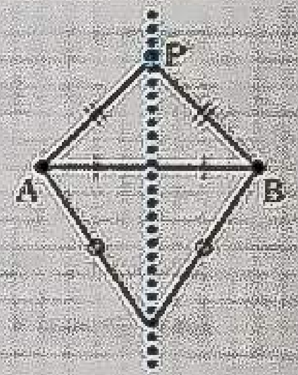
Time : to :

1. Given two points A and B as shown in the diagram, there are numerous points P where AP=BP. Draw many points such that AP=BP.



As shown above, a ***locus*** (plural: ***loci***) is a set of points that satisfies particular conditions.

Ex. Given points A(0, 2) and B(4, 0) where AP=BP, find the locus of point P.

[Sol] Let point P be (x, y).

$AP=\sqrt{x^2+(y-2)^2}$

$BP=\sqrt{(x-4)^2+y^2}$

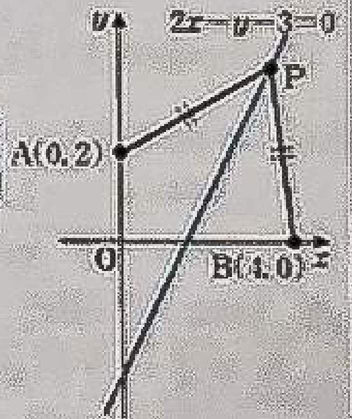
Since AP=BP,

$\sqrt{x^2+(y-2)^2}=\sqrt{(x-4)^2+y^2}$

Squaring both sides and simplifying

$\therefore\ 2x-y-3=0$

Therefore, the locus of point P is line $2x-y-3=0$.



2. Given points A(1, −1) and B(4, 4) where AP=BP, find the locus of point P

[Sol] Let point P be (x, y).

$AP=\sqrt{(x-1)^2+(y+1)^2}$

$BP=\sqrt{(x-4)^2+(y-4)^2}$

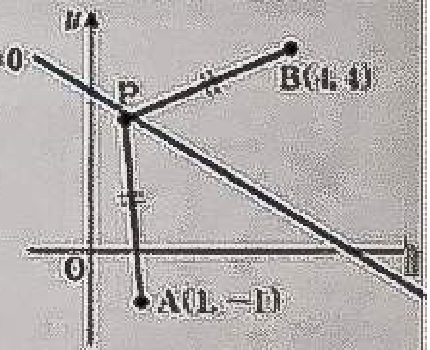
Since AP=BP,

$\sqrt{(x-1)^2+(y+1)^2}=\sqrt{(x-4)^2+(y-4)^2}$

$\therefore\ 3x+5y-15=0$

Therefore, the locus of point P is

line $3x+5y-15=0$ $\left[y=-\dfrac{3}{5}x+3\right]$.



3. Given points A$(-3, 0)$ and B$(1, 2)$ where AP=BP, find the locus of point P.

[Sol] Let point P be (x, y).

$AP=\sqrt{(x+3)^2+y^2}$

$BP=\sqrt{(x-1)^2+(y-2)^2}$

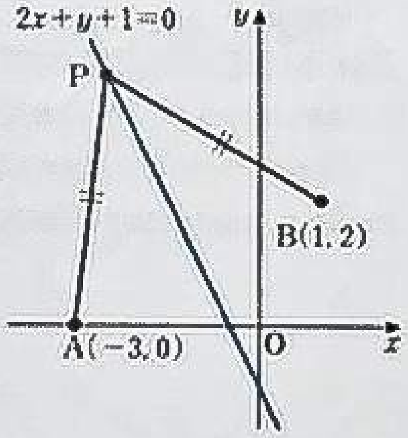
Since AP=BP,

$\sqrt{(x+3)^2+y^2}=\sqrt{(x-1)^2+(y-2)^2}$

$\therefore\ 2x+y+1=0$

Therefore, the locus of point P is

line $2x+y+1=0$ $[y=-2x-1]$.



4. Given points A$(-1, 3)$ and B$(3, 1)$ where AP=BP, find the locus of point P.

[Sol] Let point P be (x, y).

$AP=\sqrt{(x+1)^2+(y-3)^2}$

$BP=\sqrt{(x-3)^2+(y-1)^2}$

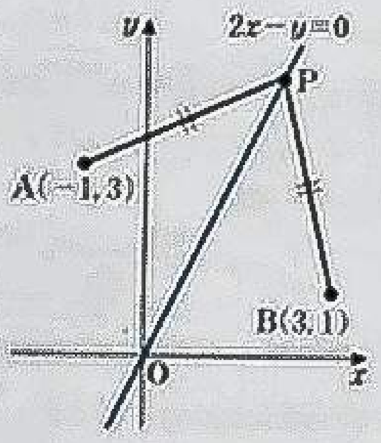
Since AP=BP,

$\sqrt{(x+1)^2+(y-3)^2}=\sqrt{(x-3)^2+(y-1)^2}$

$\therefore\ 2x-y=0$

Therefore, the locus of point P is

line $2x-y=0$ $[y=2x]$.



Note A figure (such as a line and a circle) should be written in the answer for each of the "find the locus" questions.

M52a

KUMON

Loci 1

Name

Date / /

Time : to :

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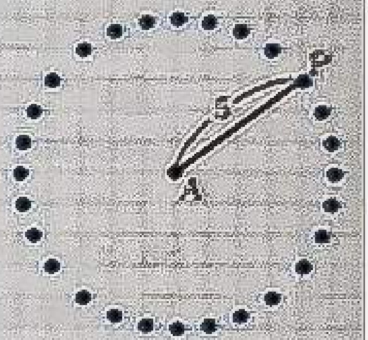
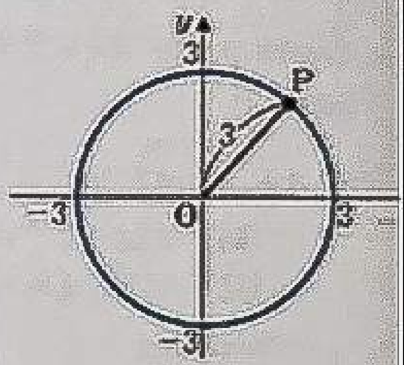
1. Given point A as shown in the diagram, point P is the point whose distance from point A is 5. There are numerous points P where AP=5. Draw many points such that AP=5.



2. Find the locus of point P such that the distance from point P to origin O is 3.



[Sol] Let point P be (x, y).

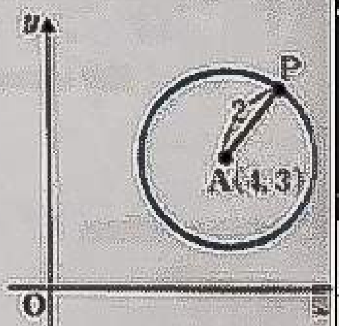
Since the distance between points O and P is 3,

$\sqrt{x^2+y^2}=\boxed{3}$

$\therefore\ x^2+y^2=\boxed{9}$ Squaring both sides

Therefore, the locus of point P is a circle with its center at point $(\boxed{0}, \boxed{0})$ and radius $\boxed{3}$.

3. Find the locus of point P such that the distance from point P to A(4, 3) is 2.



[Sol] Let point P be (x, y).

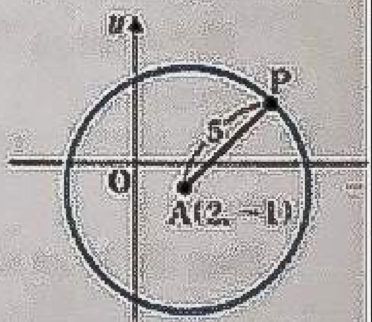
Since the distance between points A and P is 2,

$\sqrt{(x-4)^2+(y-3)^2}=2$

$\therefore\ (x-4)^2+(y-3)^2=4$

Therefore, the locus of point P is a circle with its center at point (4, 3) and radius 2.

4. Find the locus of point P such that the distance from point P to A(2, −1) is 5.



[Sol] Let point P be (x, y).

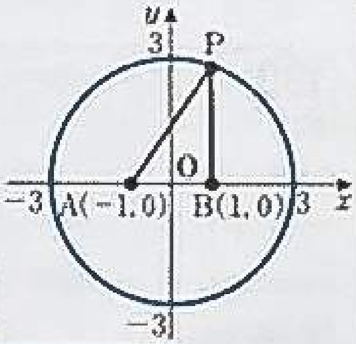
Since the distance between points A and P is 5,

$\sqrt{(x-2)^2+(y+1)^2}=5$

$\therefore\ (x-2)^2+(y+1)^2=25$

Therefore, the locus of point P is a circle with its center at point (2, −1 and radius 5.

Ex Given points $A(-1, 0)$ and $B(1, 0)$ where $AP^2+BP^2=20$, find the locus of point P.



[Sol] Let point P be (x, y).

$$AP^2=(x+1)^2+y^2$$
$$BP^2=(x-1)^2+y^2$$

Therefore,

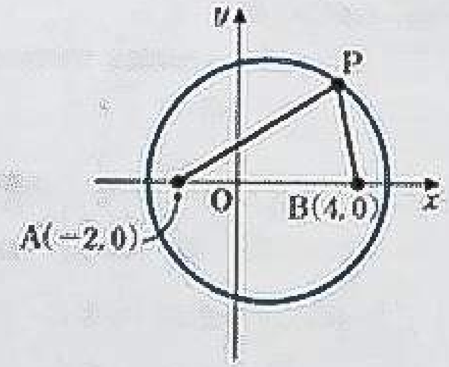
$$[(x+1)^2+y^2]+[(x-1)^2+y^2]=20$$

Substituting into $AP^2+BP^2=20$

So, $x^2+y^2=9$

Thus, the locus is a circle with its center at point $(0, 0)$ and radius 3.

5. Given points $A(-2, 0)$ and $B(4, 0)$ where $AP^2+BP^2=50$, find the locus of point P.



[Sol] Let point P be (x, y).

$$AP^2=(x+2)^2+y^2$$
$$BP^2=(x-4)^2+y^2$$

Therefore,

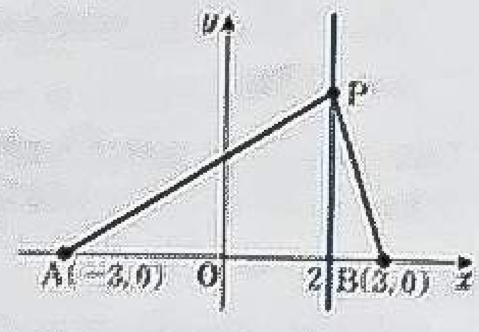
$$[(x+2)^2+y^2]+[(x-4)^2+y^2]=50$$

So, $x^2+y^2-2x-15=0$

$$(x-1)^2+y^2=16$$

Thus, the locus is a circle with its center at point $(1, 0)$ and radius 4.

Given points $A(-3, 0)$ and $B(3, 0)$ where $AP^2-BP^2=24$, find the locus of point P.



[Sol] Let point P be (x, y).

$$AP^2=(x+3)^2+y^2$$
$$BP^2=(x-3)^2+y^2$$

Therefore,

$$[(x+3)^2+y^2]-[(x-3)^2+y^2]=24$$

So, $x=2$

Thus, the locus is line $x=2$.

Loci 1

Name

Date / /

Time : to :

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1. Given points A(0, 0), B(2, 0) and C(0, 2) where $2AP^2 = BP^2 + CP^2$, find the locus of point P.

[Sol] Let point P be (x, y).

$$AP^2 = x^2 + y^2$$
$$BP^2 = (x-2)^2 + y^2$$
$$CP^2 = x^2 + (y-2)^2$$

Therefore,

$$2(x^2+y^2) = [(x-2)^2+y^2] + [x^2+(y-2)^2]$$

So, $x+y-2=0$

Thus, the locus is line $x+y-2=0$ $[y=-x+2]$.

2. Given points A(3, −2), B(1, 2) and C(5, 4) where $AP^2 + BP^2 = CP^2$, find the locus of point P.

[Sol] Let point P be (x, y).

$$AP^2 = (x-3)^2 + (y+2)^2$$
$$BP^2 = (x-1)^2 + (y-2)^2$$
$$CP^2 = (x-5)^2 + (y-4)^2$$

Therefore,

$$[(x-3)^2+(y+2)^2] + [(x-1)^2+(y-2)^2] = (x-5)^2+(y-4)^2$$

So, $x^2+y^2+2x+8y-23=0$

$$(x+1)^2+(y+4)^2=40$$

Thus, the locus is a circle with its center at point $(-1, -4)$ and radius $2\sqrt{10}$.

M53b

3. Given points A(0, 0), B(3, 0) and C(0, 3) where $BP^2 - AP^2 = 3CP^2$, find the locus of point P.

[Sol] Let point P be (x, y).

$$AP^2 = x^2 + y^2$$
$$BP^2 = (x-3)^2 + y^2$$
$$CP^2 = x^2 + (y-3)^2$$

Therefore,

$$[(x-3)^2 + y^2] - (x^2 + y^2) = 3[x^2 + (y-3)^2]$$

So, $x^2 + y^2 + 2x - 6y + 6 = 0$

$$(x+1)^2 + (y-3)^2 = 4$$

Thus, the locus is a circle with its center at point $(-1, 3)$ and radius 2.

4. Given points A(−5, 0), B(5, 0) and C(3, 6) where $AP^2 + BP^2 + CP^2 = 170$, find the locus of point P.

[Sol] Let point P be (x, y).

$$AP^2 = (x+5)^2 + y^2$$
$$BP^2 = (x-5)^2 + y^2$$
$$CP^2 = (x-3)^2 + (y-6)^2$$

Therefore,

$$[(x+5)^2 + y^2] + [(x-5)^2 + y^2] + [(x-3)^2 + (y-6)^2] = 170$$

So, $x^2 + y^2 - 2x - 4y - 25 = 0$

$$(x-1)^2 + (y-2)^2 = 30$$

Thus, the locus is a circle with its center at point $(1, 2)$ and radius $\sqrt{30}$.

Loci 1

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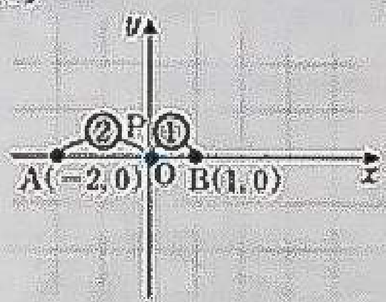
100%	~90%	~80%	~70%	69%~

Given points A$(-2, 0)$ and B$(1, 0)$ where AP : BP$=2:1$, find the locus of point P. Fill in the following blanks.

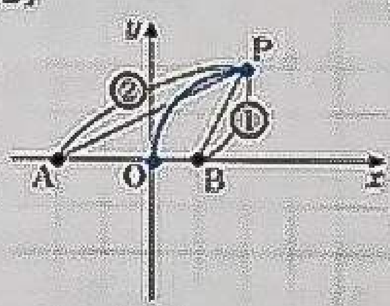
[Sol] Moving point P while satisfying AP : BP$=2:1$, the locus of point P can be found as follows:

From the diagrams [1]~[5] below, the locus is a $\boxed{\text{circle}}$ with its center at point ($\boxed{2}$, 0) and radius $\boxed{2}$.

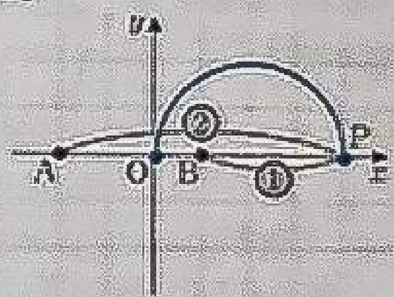
[1]



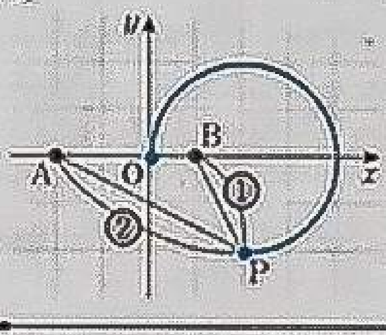
[2]



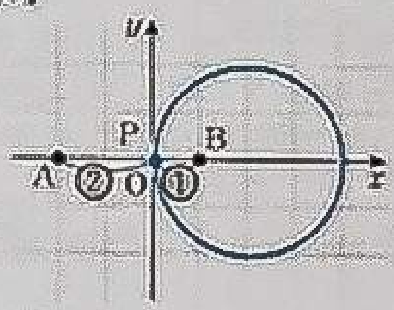
[3]



[4]

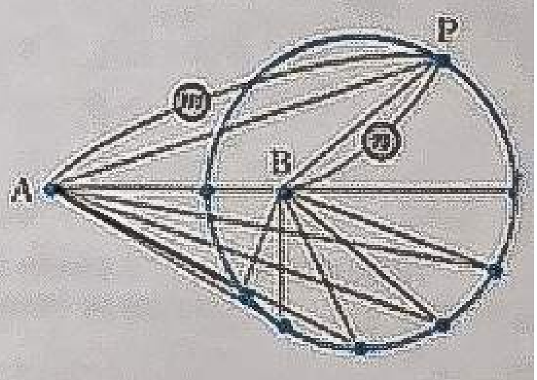


[5]



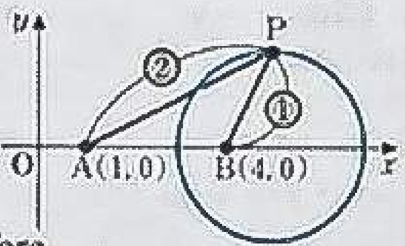
nswers: circle, 2, 2

Given two fixed points A and B, the locus of point P which satisfies AP : BP$=m:n\ (m \neq n)$ becomes a circle whose endpoints of its diameter are the points which internally and externally divide line segment AB in the ratio $m:n$. This type of circle is called the *circle of Apollonius*.



M54b

EX Given points A(1, 0) and B(4, 0) where AP : BP = 2 : 1, find the locus of point P.



[Sol] Let point P be (x, y).

$AP=\sqrt{(x-1)^2+y^2}$

$BP=\sqrt{(x-4)^2+y^2}$

Since AP : BP = 2 : 1, AP = 2BP; therefore,

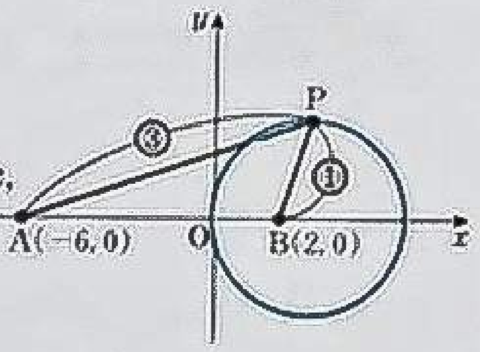
$\sqrt{(x-1)^2+y^2}=2\sqrt{(x-4)^2+y^2}$ Squaring both sides and simplifying

$\therefore\ x^2+y^2-10x+21=0$

$(x-5)^2+y^2=4$

Thus, the locus is a circle with its center at point (5, 0) and radius 2.

1. Given points A(−6, 0) and B(2, 0) where AP : BP = 3 : 1, find the locus of point P.



[Sol] Let point P be (x, y).

$AP=\sqrt{(x+6)^2+y^2}$

$BP=\sqrt{(x-2)^2+y^2}$

Since AP : BP = 3 : 1, AP = 3BP; therefore,

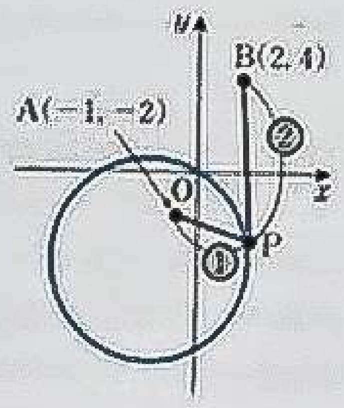
$\sqrt{(x+6)^2+y^2}=3\sqrt{(x-2)^2+y^2}$

$\therefore\ x^2+y^2-6x=0$

$(x-3)^2+y^2=9$

Thus, the locus is a circle with its center at point (3, 0) and radius 3.

2. Given points A(−1, −2) and B(2, 4) where AP : BP = 1 : 2, find the locus of point P.



[Sol] Let point P be (x, y).

$AP=\sqrt{(x+1)^2+(y+2)^2}$

$BP=\sqrt{(x-2)^2+(y-4)^2}$

Since AP : BP = 1 : 2, 2AP = BP; therefore,

$2\sqrt{(x+1)^2+(y+2)^2}=\sqrt{(x-2)^2+(y-4)^2}$

$\therefore\ x^2+y^2+4x+8y=0$

$(x+2)^2+(y+4)^2=20$

Thus, the locus is a circle with its center at point (−2, −4) and radius $2\sqrt{5}$.

Loci 1

Name

Date / /

Time : to :

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Ex. Given moving point Q on circle $x^2+y^2=4$, find the locus of midpoint P of line segment AQ connecting point A(8, 0) and point Q.

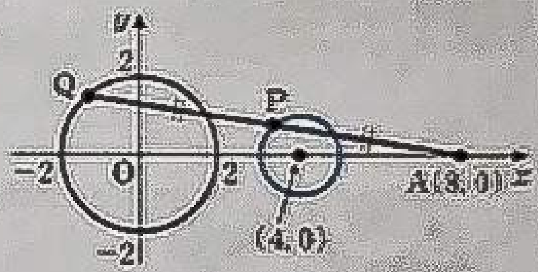
[Sol] Let point P be (x, y) and point Q be (s, t). Since point Q lies on circle $x^2+y^2=4$,

$s^2+t^2=4$ …①

Since point P is the midpoint of line segment AQ,

$x=\frac{8+s}{2}$, i.e. $s=2x-8$

$y=\frac{0+t}{2}$, i.e. $t=2y$



Substituting them into ①,

$(2x-8)^2+(2y)^2=4$

So, $(x-4)^2+y^2=1$

Therefore, the locus is a circle with its center at point (4, 0) and radius 1.

1. Given moving point Q on circle $x^2+y^2=16$, find the locus of midpoint P of line segment AQ connecting point A(6, 0) and point Q.

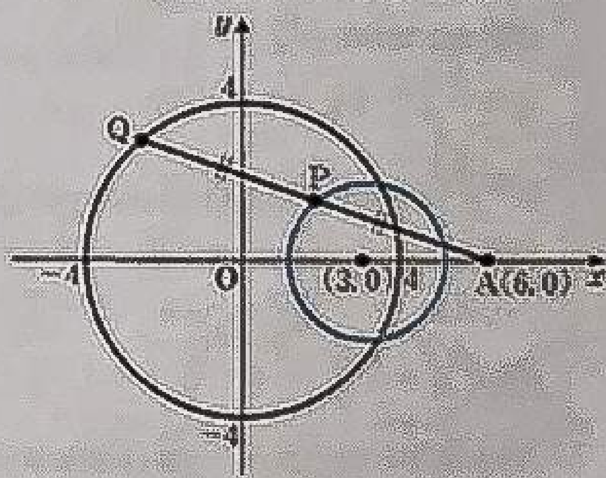
[Sol] Let point P be (x, y) and point Q be (s, t). Since point Q lies on circle $x^2+y^2=16$,

$s^2+t^2=16$ …①

Since point P is the midpoint of line segment AQ,

$x=\frac{6+s}{2}$, i.e. $s=2x-6$

$y=\frac{0+t}{2}$, i.e. $t=2y$



Substituting them into ①,

$(2x-6)^2+(2y)^2=16$

So, $(x-3)^2+y^2=4$

Therefore, the locus is a circle with its center at point (3, 0) and radius 2

M55b

2. Given moving point Q on circle $x^2+y^2=9$, find the locus of midpoint P of line segment AQ connecting point A(6, 4) and point Q.

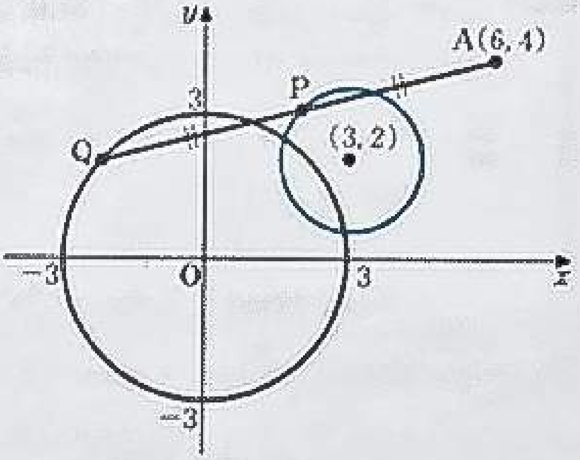
[Sol] Let point P be (x, y) and point Q be (s, t). Since point Q lies on circle $x^2+y^2=9$,

$$s^2+t^2=9 \quad \cdots ①$$

Since point P is the midpoint of line segment AQ,

$x=\dfrac{6+s}{2}$, i.e. $s=2x-6$

$y=\dfrac{4+t}{2}$, i.e. $t=2y-4$



Substituting them into ①,

$$(2x-6)^2+(2y-4)^2=9$$

So, $(x-3)^2+(y-2)^2=\dfrac{9}{4}$

Therefore, **the locus is a circle with its center at point (3, 2) and radius $\dfrac{3}{2}$.**

3. Given moving point Q on circle $x^2+y^2+4x-2y+1=0$, find the locus of midpoint P of line segment AQ connecting point A(3, 2) and point Q.

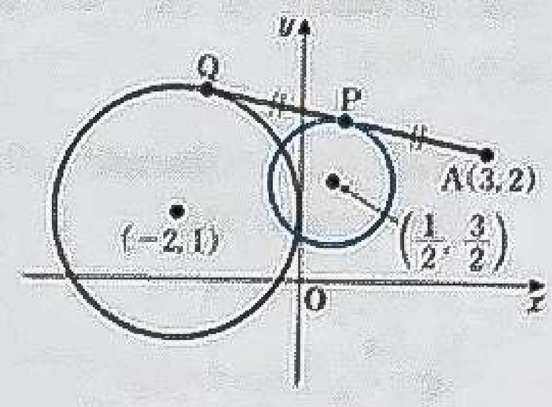
[Sol] Let point P be (x, y) and point Q be (s, t). Since point Q lies on circle $x^2+y^2+4x-2y+1=0$, i.e. $(x+2)^2+(y-1)^2=4$,

$$(s+2)^2+(t-1)^2=4 \quad \cdots ①$$

Since point P is the midpoint of line segment AQ,

$x=\dfrac{3+s}{2}$, i.e. $s=2x-3$

$y=\dfrac{2+t}{2}$, i.e. $t=2y-2$



Substituting them into ①,

$$[(2x-3)+2]^2+[(2y-2)-1]^2=4$$

So, $\left(x-\dfrac{1}{2}\right)^2+\left(y-\dfrac{3}{2}\right)^2=1$

Therefore, **the locus is a circle with its center at point $\left(\dfrac{1}{2}, \dfrac{3}{2}\right)$ and radius 1.**

Loci 1

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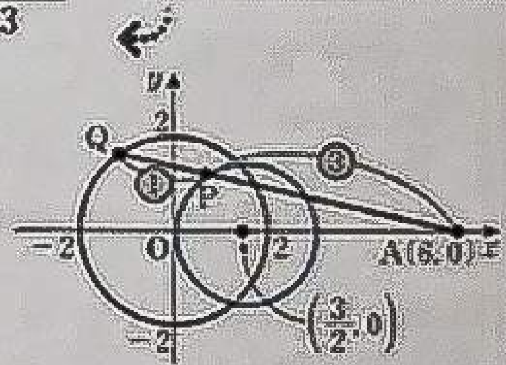
Ex. Given moving point Q on circle $x^2+y^2=4$, find the locus of point P internally dividing in the ratio 3 : 1 the line segment AQ connecting point A$(6, 0)$ and point Q.

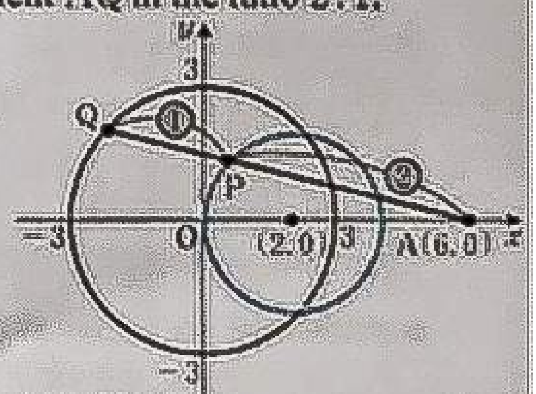
> From Coordinates of Internal Dividing Points (M6): given points $A(x_1, y_1)$ and $B(x_2, y_2)$, the coordinates of the point that internally divides line segment AB in the ratio $m : n$ are $\left(\frac{nx_1+mx_2}{m+n}, \frac{ny_1+my_2}{m+n}\right)$

[Sol] Let point P be (x, y) and point Q be (s, t).
Since point Q lies on circle $x^2+y^2=4$,
$s^2+t^2=4$ …①
Since point P internally divides line segment AQ in the ratio 3 : 1,
$x=\frac{1\cdot6+3\cdot s}{3+1}=\frac{3s+6}{4}$, i.e. $s=\frac{4x-6}{3}$
$y=\frac{1\cdot0+3\cdot t}{3+1}=\frac{3t}{4}$, i.e. $t=\frac{4}{3}y$
Substituting them into ①,
$$\left(\frac{4x-6}{3}\right)^2+\left(\frac{4}{3}y\right)^2=4$$
So, $\left(x-\frac{3}{2}\right)^2+y^2=\frac{9}{4}$
Therefore, the locus is
a circle with its center at point $\left(\frac{3}{2}, 0\right)$ and radius $\frac{3}{2}$.



1. Given moving point Q on circle $x^2+y^2=9$, find the locus of point P internally dividing in the ratio 2 : 1 the line segment AQ connecting point A$(6, 0)$ and point Q.

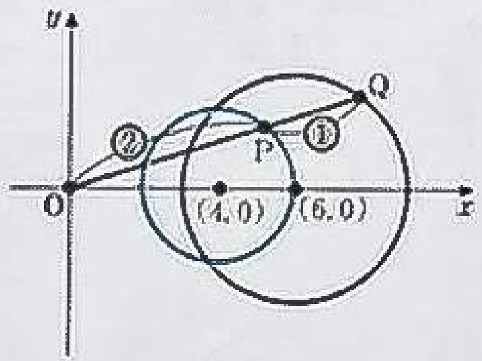
[Sol] Let point P be (x, y) and point Q be (s, t).
Since point Q lies on circle $x^2+y^2=9$,
$s^2+t^2=9$ …①
Since point P internally divides line segment AQ in the ratio 2 : 1,
$x=\frac{1\cdot6+2\cdot s}{2+1}=\frac{2s+6}{3}$, i.e. $s=\frac{3x-6}{2}$
$y=\frac{1\cdot0+2\cdot t}{2+1}=\frac{2t}{3}$, i.e. $t=\frac{3}{2}y$
Substituting them into ①,
$$\left(\frac{3x-6}{2}\right)^2+\left(\frac{3}{2}y\right)^2=9$$
So, $(x-2)^2+y^2=4$
Therefore, the locus is
a circle with its center at point $(2, 0)$ and radius 2.



M56b

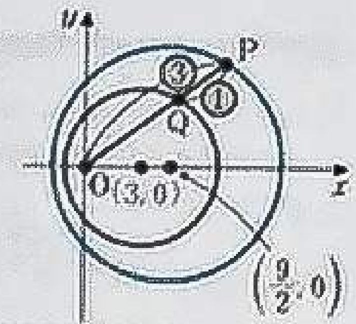
2. Given moving point Q on circle $(x-6)^2+y^2=9$, find the locus of point P internally dividing in the ratio 2 : 1 the line segment OQ connecting origin O and point Q.

[Sol] Let point P be (x, y) and point Q be (s, t).
Since point Q lies on circle $(x-6)^2+y^2=9$,
$(s-6)^2+t^2=9$ ...①
Since point P internally divides line segment OQ in the ratio 2 : 1,
$x=\frac{1\cdot 0+2\cdot s}{2+1}=\frac{2s}{3}$, i.e. $s=\frac{3}{2}x$
$y=\frac{1\cdot 0+2\cdot t}{2+1}=\frac{2t}{3}$, i.e. $t=\frac{3}{2}y$
Substituting them into ①,
$\left(\frac{3}{2}x-6\right)^2+\left(\frac{3}{2}y\right)^2=9$
So, $(x-4)^2+y^2=4$
Therefore, the locus is
a circle with its center at point (4, 0) and radius 2.



3. Given moving point Q on circle $(x-3)^2+y^2=16$, find the locus of point P externally dividing in the ratio 3 : 1 the line segment OQ connecting origin O and point Q.

[Sol] Let point P be (x, y) and point Q be (s, t).
Since point Q lies on circle $(x-3)^2+y^2=16$,
$(s-3)^2+t^2=16$ ...①
Since point P externally divides line segment OQ in the ratio 3 : 1,
$x=\frac{-1\cdot 0+3\cdot s}{3-1}=\frac{3s}{2}$, i.e. $s=\frac{2}{3}x$
$y=\frac{-1\cdot 0+3\cdot t}{3-1}=\frac{3t}{2}$, i.e. $t=\frac{2}{3}y$
Substituting them into ①,
$\left(\frac{2}{3}x-3\right)^2+\left(\frac{2}{3}y\right)^2=16$
So, $\left(x-\frac{9}{2}\right)^2+y^2=36$
Therefore, the locus is
a circle with its center at point $\left(\frac{9}{2}, 0\right)$ and radius 6.



From Coordinates of External Dividing Points (M6), given points $A(x_1, y_1)$ and $B(x_2, y_2)$, the coordinates of the point that externally divides line segment AB in the ratio $m:n$ are
$\left(\frac{-nx_1+mx_2}{m-n}, \frac{-ny_1+my_2}{m-n}\right)$

Loci 1

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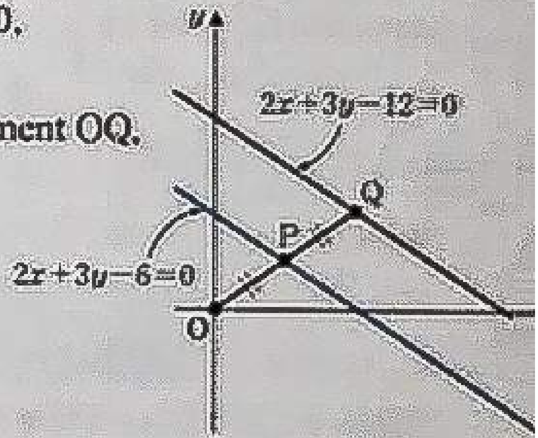
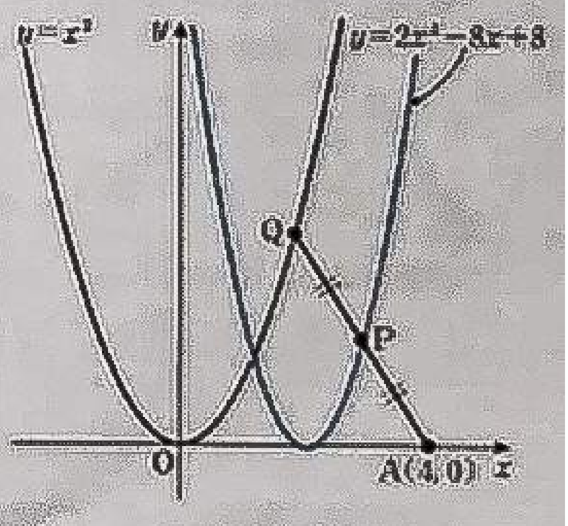
1. Given moving point Q on line $2x+3y-12=0$, find the locus of midpoint P of line segment OQ connecting origin O and point Q.

[Sol] Let point P be (x, y) and point Q be (s, t).
Since point Q lies on line $2x+3y-12=0$,
$2s+3t-12=0 \quad \cdots①$
Since point P is the midpoint of line segment OQ,
$x=\dfrac{0+s}{2}$, i.e. $s=2x$
$y=\dfrac{0+t}{2}$, i.e. $t=2y$
Substituting them into ①,
$2\cdot 2x+3\cdot 2y-12=0$
So, $2x+3y-6=0$
Therefore, the locus is line $2x+3y-6=0$ $\left[y=-\dfrac{2}{3}x+2\right]$.



2. Given moving point Q on parabola $y=x^2$, find the locus of midpoint P of line segment AQ connecting point A(4, 0) and point Q.

[Sol] Let point P be (x, y) and point Q be (s, t).
Since point Q lies on parabola $y=x^2$,
$t=s^2 \quad \cdots①$
Since point P is the midpoint of line segment AQ,
$x=\dfrac{4+s}{2}$, i.e. $s=2x-4$
$y=\dfrac{0+t}{2}$, i.e. $t=2y$
Substituting them into ①,
$2y=(2x-4)^2$
So, $y=2x^2-8x+8$
Therefore, the locus is parabola $y=2x^2-8x+8$.



157b

Given moving point Q on line $2x+y+1=0$, find the locus of point P internally dividing in the ratio 2 : 1 the line segment AQ connecting point A(3, 1) and point Q.

Sol] Let point P be (x, y) and point Q be (s, t).

Since point Q lies on line $2x+y+1=0$,

$2s+t+1=0$ …①

Since point P internally divides line segment AQ in the ratio 2 : 1,

$x=\frac{1\cdot 3+2\cdot s}{2+1}=\frac{2s+3}{3}$, i.e. $s=\frac{3x-3}{2}$

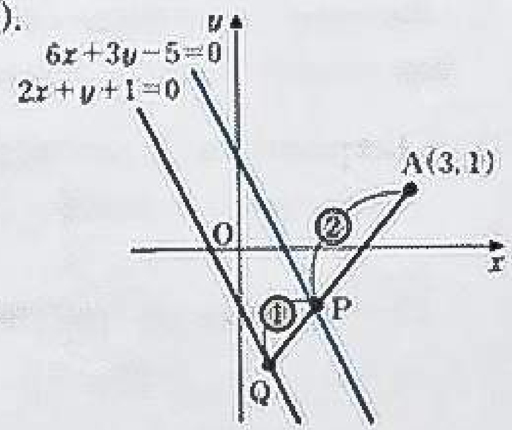
$y=\frac{1\cdot 1+2\cdot t}{2+1}=\frac{2t+1}{3}$, i.e. $t=\frac{3y-1}{2}$

Substituting them into ①,

$2\cdot\frac{3x-3}{2}+\frac{3y-1}{2}+1=0$

So, $6x+3y-5=0$

Therefore, the locus is **line $6x+3y-5=0$** $\left[y=-2x+\frac{5}{3}\right]$.



Given moving point Q on parabola $y=x^2-2x+4$, find the locus of point P externally dividing in the ratio 2 : 3 the line segment AQ connecting point A(2, 2) and point Q.

Sol] Let point P be (x, y) and point Q be (s, t).

Since point Q lies on parabola $y=x^2-2x+4$,

$t=s^2-2s+4$ …①

Since point P externally divides line segment AQ in the ratio 2 : 3,

$x=\frac{-3\cdot 2+2\cdot s}{2-3}=-2s+6$, i.e. $s=\frac{-x+6}{2}$

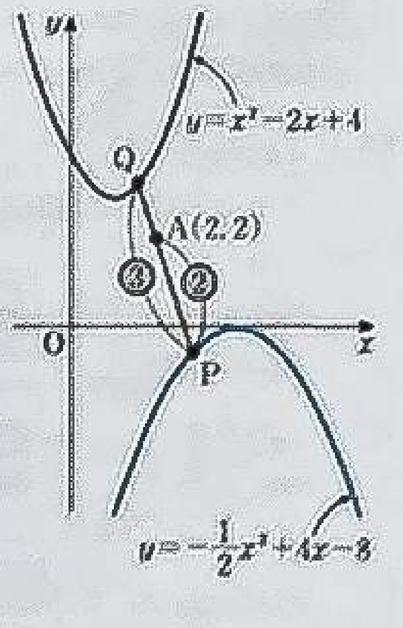
$y=\frac{-3\cdot 2+2\cdot t}{2-3}=-2t+6$, i.e. $t=\frac{-y+6}{2}$

Substituting them into ①,

$\frac{-y+6}{2}=\left(\frac{-x+6}{2}\right)^2-2\cdot\left(\frac{-x+6}{2}\right)+4$

So, $y=-\frac{1}{2}x^2+4x-8$

Therefore, the locus is **parabola $y=-\frac{1}{2}x^2+4x-8$**.



M58a

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Loci 1

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1. Given moving point Q on circle $x^2+y^2=9$, find the locus of the center of gravity G of $\triangle ABQ$ with vertices A(0, 6), B(9, 0) and Q.

[Sol] Let point G be (x, y) and point Q be (s, t).

Since point Q lies on circle $x^2+y^2=9$,

$s^2+t^2=9$ ⋯①

Since point G is the center of gravity of $\triangle ABQ$,

$x=\dfrac{0+9+s}{3}=\dfrac{s+9}{3}$, i.e. $s=3x-9$

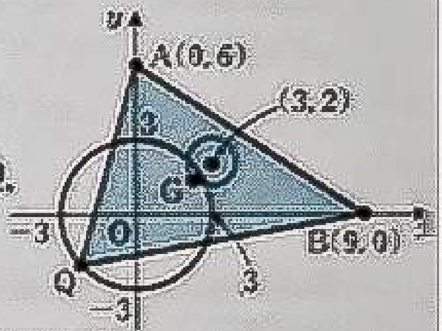
$y=\dfrac{6+0+t}{3}=\dfrac{t+6}{3}$, i.e. $t=3y-6$

Substituting them into ①,

$(3x-9)^2+(3y-6)^2=9$

So, $(x-3)^2+(y-2)^2=1$

Therefore, the locus is a circle with its center at point (3, 2) and radius 1



Center of Gravity of Triangles (M8)

2. Given moving point Q on circle $x^2+y^2=1$, find the locus of the center of gravity G of $\triangle ABQ$ with vertices A(4, 0), B(2, 3) and Q.

[Sol] Let point G be (x, y) and point Q be (s, t).

Since point Q lies on circle $x^2+y^2=1$,

$s^2+t^2=1$ ⋯①

Since point G is the center of gravity of $\triangle ABQ$,

$x=\dfrac{4+2+s}{3}=\dfrac{s+6}{3}$, i.e. $s=3x-6$

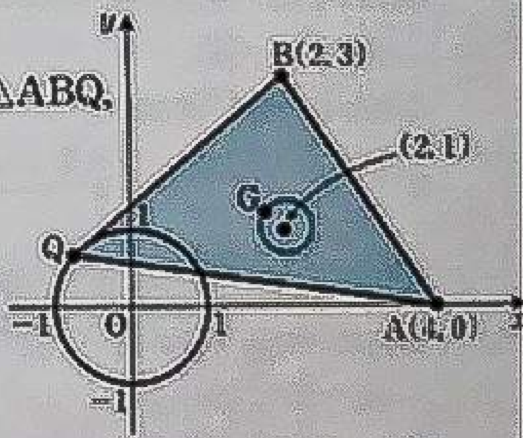
$y=\dfrac{0+3+t}{3}=\dfrac{t+3}{3}$, i.e. $t=3y-3$

Substituting them into ①,

$(3x-6)^2+(3y-3)^2=1$

So, $(x-2)^2+(y-1)^2=\dfrac{1}{9}$

Therefore, the locus is a circle with its center at point (2, 1) and radius $\dfrac{1}{3}$.

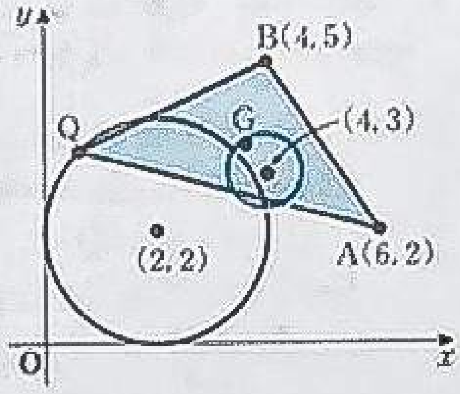


M158b

. Given moving point Q on circle $(x-2)^2+(y-2)^2=4$, find the locus of the center of gravity G of $\triangle ABQ$ with vertices A(6, 2), B(4, 5) and Q.

Sol] Let point G be (x, y) and point Q be (s, t). Since point Q lies on circle $(x-2)^2+(y-2)^2=4$,

$(s-2)^2+(t-2)^2=4$ …①



Since point G is the center of gravity of $\triangle ABQ$,

$x=\dfrac{6+4+s}{3}=\dfrac{s+10}{3}$, i.e. $s=3x-10$

$y=\dfrac{2+5+t}{3}=\dfrac{t+7}{3}$, i.e. $t=3y-7$

Substituting them into ①,

$\{(3x-10)-2\}^2+\{(3y-7)-2\}^2=4$

So, $(x-4)^2+(y-3)^2=\dfrac{4}{9}$

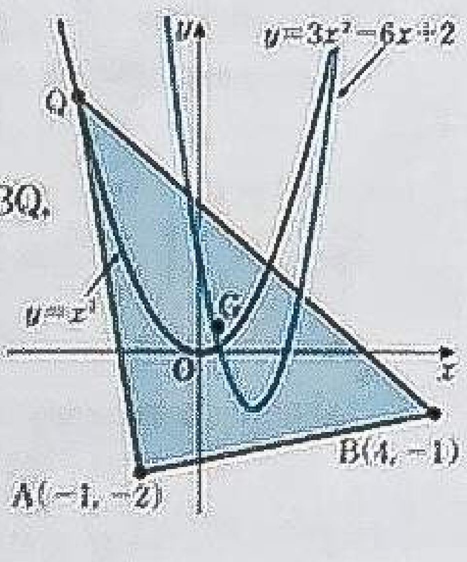
Therefore, the locus is a circle with its center at point (4, 3) and radius $\dfrac{2}{3}$.

. Given moving point Q on parabola $y=x^2$, find the locus of the center of gravity G of $\triangle ABQ$ with vertices A(−1, −2), B(4, −1) and Q.

Sol] Let point G be (x, y) and point Q be (s, t).

Since point Q lies on parabola $y=x^2$,

$t=s^2$ …①



Since point G is the center of gravity of $\triangle ABQ$,

$x=\dfrac{-1+4+s}{3}=\dfrac{s+3}{3}$, i.e. $s=3x-3$

$y=\dfrac{-2-1+t}{3}=\dfrac{t-3}{3}$, i.e. $t=3y+3$

Substituting them into ①,

$3y+3=(3x-3)^2$

So, $y=3x^2-6x+2$

Therefore, the locus is parabola $y=3x^2-6x+2$.

Name

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Time : to :

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1. Let point P be the point symmetric to point Q(s, t) with respect to line $y=2x$.

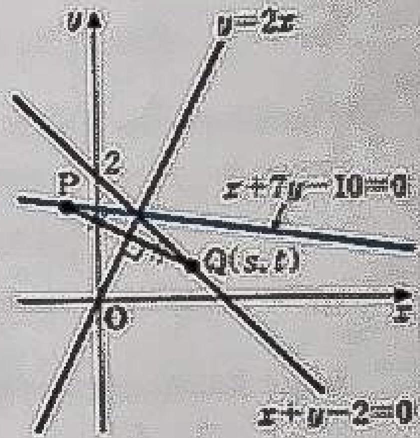
(1) Express the coordinates of point P in terms of s and t.

[Sol] Let point P be (x, y). Since the slope of $y=2x$ is 2 and line PQ is perpendicular to $y=2x$,

$$2\cdot\frac{t-y}{s-x}=-1$$

So, $x+2y=s+2t$ ···①

Also, the midpoint $\left(\frac{x+s}{2}, \frac{y+t}{2}\right)$ of line segment PQ lies on $y=2x$,



$$\frac{y+t}{2}=2\cdot\frac{x+s}{2}$$

So, $2x-y=-2s+t$ ···②

From ① and ②,

$$x=\frac{-3s+4t}{5},\ y=\frac{4s+3t}{5}$$ ← Solving ① and ② for x and y

$$\therefore\ \mathrm{P}\left(\frac{-3s+4t}{5}, \frac{4s+3t}{5}\right)$$

(2) Given the condition in question (1) and the moving point Q on line $x+y-2=0$, find the locus of point P.

[Sol] Since point Q(s, t) lies on $x+y-2=0$,

$s+t-2=0$ ···③

From ① and ②,

$$s=\frac{-3x+4y}{5},\ t=\frac{4x+3y}{5}$$ ← Solving ① and ② for s and t

Substituting them into ③,

$$\frac{-3x+4y}{5}+\frac{4x+3y}{5}-2=0$$

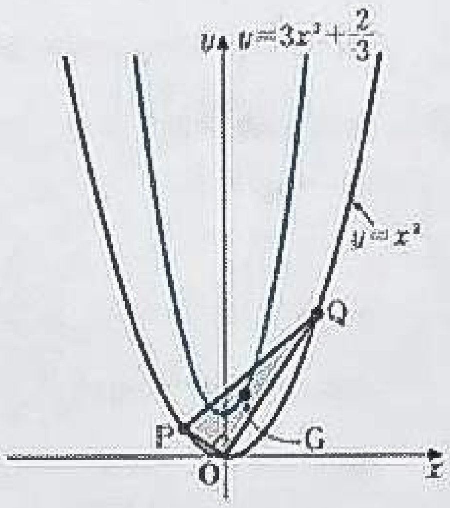
So, $x+7y-10=0$

Therefore, the locus is line $x+7y-10=0$ $\left[y=-\frac{1}{7}x+\frac{10}{7}\right]$.

M59b

. Place two points P and Q on parabola $y=x^2$, and connect P, Q and origin O such that $\angle POQ$ is a right angle.

1) Given that the coordinates of point P are (t, t^2), express the coordinates of point Q in terms of t.



Sol] From $\angle POQ=90°$, since points P and Q do not coincide with origin O, $t\neq 0$

Since the slope of line OP is $\frac{t^2-0}{t-0}=t$,

the slope of line OQ is $-\frac{1}{t}$.

Therefore, line OQ is $y=-\frac{1}{t}x$.

Since the point of intersection of this line and $y=x^2$ is $-\frac{1}{t}x=x^2$, $x\left(x+\frac{1}{t}\right)=0$

$\therefore\ x=0,\ -\frac{1}{t}$

Since it coincides with the origin when $x=0$,

$Q\left(-\frac{1}{t}, \frac{1}{t^2}\right)$

2) When point P moves along the parabola, find the locus of the center of gravity of $\triangle POQ$.

Sol] From question (1), let the center of gravity of $\triangle POQ$ be $G(x, y)$.

$$x=\frac{t+0-\frac{1}{t}}{3}=\frac{1}{3}\left(t-\frac{1}{t}\right) \quad \cdots①$$

$$y=\frac{t^2+0+\frac{1}{t^2}}{3}=\frac{1}{3}\left(t^2+\frac{1}{t^2}\right) \quad \cdots②$$

From points $P(t, t^2)$, $O(0, 0)$ and $Q\left(-\frac{1}{t}, \frac{1}{t^2}\right)$

From ①, $x^2=\left[\frac{1}{3}\left(t-\frac{1}{t}\right)\right]^2$

Squaring both sides of ①

$$=\frac{1}{9}\left(t^2+\frac{1}{t^2}-2\right)$$

Multiplying both sides by 9, $9x^2=t^2+\frac{1}{t^2}-2$

$\therefore\ t^2+\frac{1}{t^2}=9x^2+2 \quad \cdots③$

Substituting ③ into ②,

$$y=\frac{1}{3}(9x^2+2)=3x^2+\frac{2}{3}$$

Therefore, the locus is **parabola $y=3x^2+\frac{2}{3}$**.

Loci 1

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Date / /

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1. Given points A(2, 0) and B(0, 4) where $AP^2+BP^2=20$, find the locus of point P.

➡ M52

[Sol] Let point P be (x, y).

$AP^2=(x-2)^2+y^2$

$BP^2=x^2+(y-4)^2$

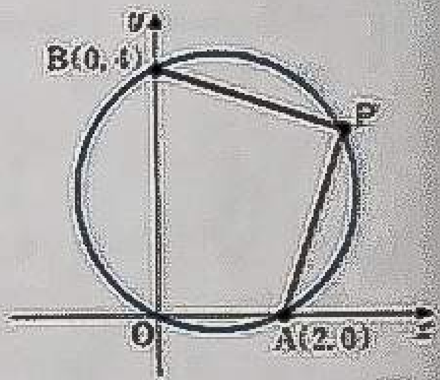
Therefore,

$[(x-2)^2+y^2]+[x^2+(y-4)^2]=20$

So, $x^2+y^2-2x-4y=0$

$(x-1)^2+(y-2)^2=5$

Thus, the locus is a circle with its center at point (1, 2) and radius $\sqrt{5}$.



2. Given points A(−4, 0) and B(5, 0) where AP : BP = 2 : 1, find the locus of point P.

➡ M54

[Sol] Let point P be (x, y).

$AP=\sqrt{(x+4)^2+y^2}$

$BP=\sqrt{(x-5)^2+y^2}$

Since AP : BP = 2 : 1, AP = 2BP;

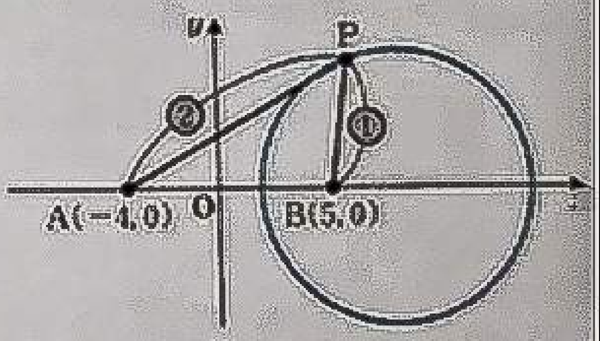
therefore,

$\sqrt{(x+4)^2+y^2}=2\sqrt{(x-5)^2+y^2}$

$\therefore\ x^2+y^2-16x+28=0$

$(x-8)^2+y^2=36$

Thus, the locus is a circle with its center at point (8, 0) and radius 6.



M60b

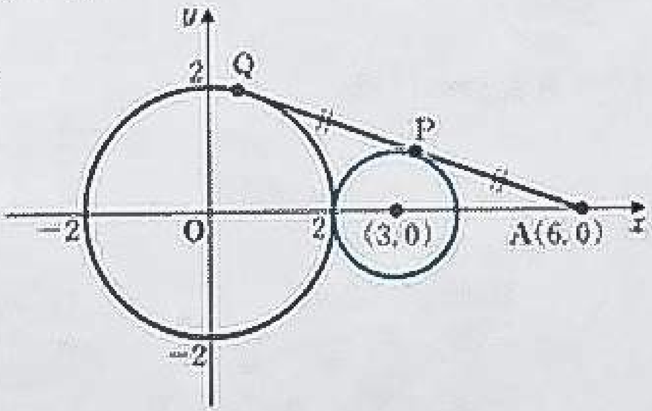
. Given moving point Q on circle $x^2+y^2=4$, find the locus of midpoint P of line segment AQ connecting point A(6, 0) and point Q. ➡ M55

Sol] Let point P be (x, y) and point Q be (s, t).

Since point Q lies on circle $x^2+y^2=4$,

$s^2+t^2=4$ ⋯①

Since point P is the midpoint of line segment AQ,



$x=\dfrac{6+s}{2}$, i.e. $s=2x-6$

$y=\dfrac{0+t}{2}$, i.e. $t=2y$

Substituting them into ①,

$(2x-6)^2+(2y)^2=4$

So, $(x-3)^2+y^2=1$

Therefore, the locus is a circle with its center at point (3, 0) and radius 1.

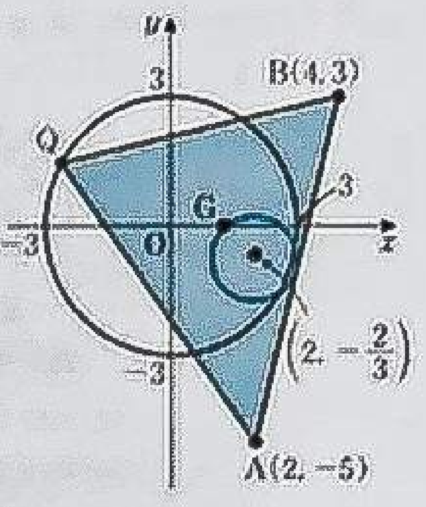
. Given moving point Q on circle $x^2+y^2=9$, find the locus of the center of gravity G of △ABQ with vertices A(2, −5), B(4, 3) and Q. ➡ M58

Sol] Let point G be (x, y) and point Q be (s, t).

Since point Q lies on circle $x^2+y^2=9$,

$s^2+t^2=9$ ⋯①

Since point G is the center of gravity of △ABQ,



$x=\dfrac{2+4+s}{3}=\dfrac{s+6}{3}$, i.e. $s=3x-6$

$y=\dfrac{-5+3+t}{3}=\dfrac{t-2}{3}$, i.e. $t=3y+2$

Substituting them into ①,

$(3x-6)^2+(3y+2)^2=9$

So, $(x-2)^2+\left(y+\dfrac{2}{3}\right)^2=1$

Therefore, the locus is a circle with its center at point $\left(2, -\dfrac{2}{3}\right)$ and radius 1.

M61a

Loci 2

Name

Date / /

Time : to :

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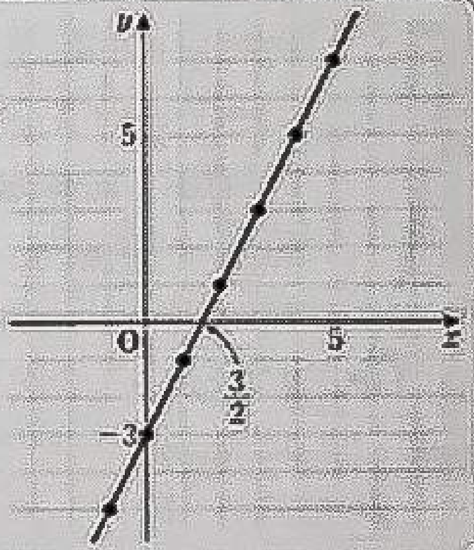
1. When the value of t changes, graph the locus of the following points $P(x, y)$.

Ex. $P(t+1, 2t-1)$

[Sol]

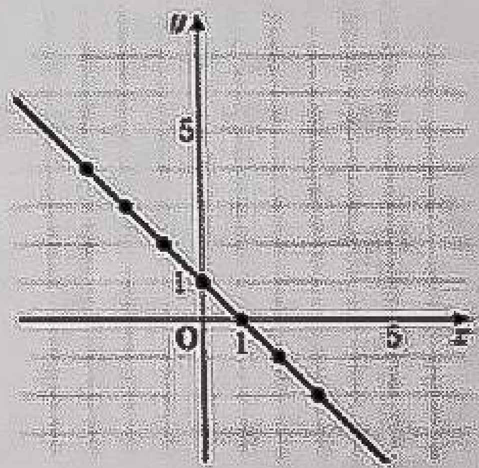
t	-2	-1	0	1	2	3	4
x	-1	0	1	2	3	4	5
y	-5	-3	-1	1	3	5	7



(1) $P(t-1, -t+2)$

[Sol]

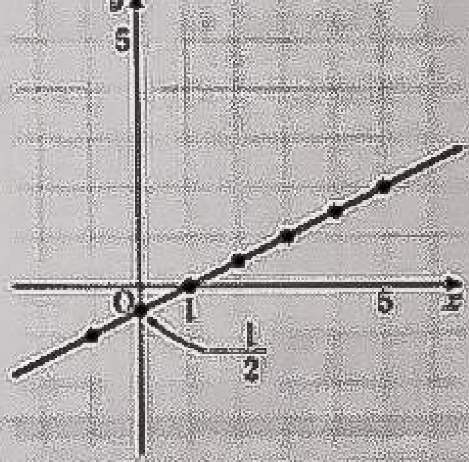
t	-2	-1	0	1	2	3	4
x	-3	-2	-1	0	1	2	3
y	4	3	2	1	0	-1	-2



(2) $P\left(t+1, \frac{1}{2}t\right)$

[Sol]

t	-2	-1	0	1	2	3	4
x	-1	0	1	2	3	4	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2



. For all real numbers t, find the locus of the following points $P(x, y)$.

Ex $P(2t-4, -t+5)$

[Sol] $x=2t-4$ ⋯①

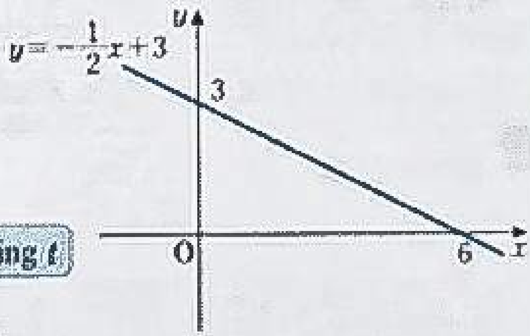
$y=-t+5$ ⋯②

From ① and ②,

$y=-\frac{1}{2}x+3$ ⟵ Eliminating t

Therefore, the locus is

line $y=-\frac{1}{2}x+3$.



1) $P(t+1, 3t+2)$

[Sol] $x=t+1$ ⋯①

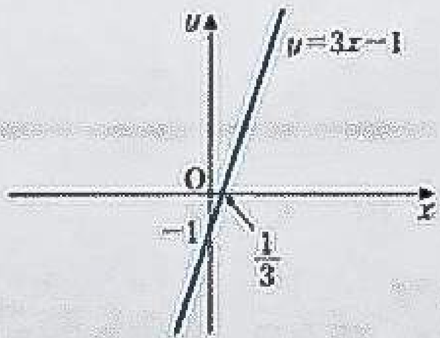
$y=3t+2$ ⋯②

From ① and ②,

$y=3x-1$

Therefore, the locus is

line $y=3x-1$ $[3x-y-1=0]$.



2) $P(-12t-6, -4t-1)$

[Sol] $x=-12t-6$ ⋯①

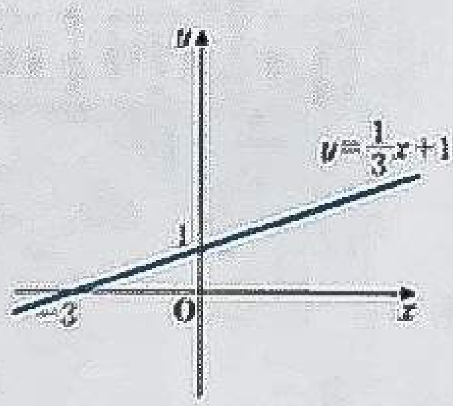
$y=-4t-1$ ⋯②

From ① and ②,

$y=\frac{1}{3}x+1$

Therefore, the locus is

line $y=\frac{1}{3}x+1$ $[x-3y+3=0]$.



Generally, when a curve on a plane is represented by a variable t such that $x=f(t)$, $y=g(t)$, then the representation is called a ***parametric representation***, and variable t is called a ***parameter***.

Loci 2

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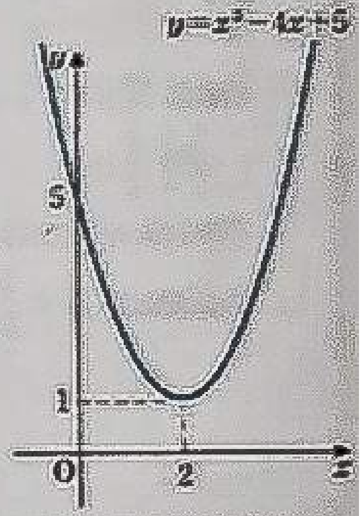
1. For all real numbers t, find the locus of the following points $P(x, y)$.

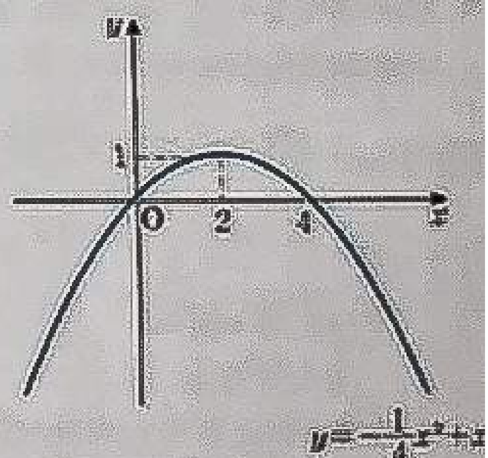
(1) $P(t+2, t^2+1)$

[Sol] $x=t+2 \quad \cdots ①$
$y=t^2+1 \quad \cdots ②$
From ① and ②,
$$y=x^2-4x+5$$
Therefore, the locus is
parabola $y=x^2-4x+5$.



(2) $P(2t, 2t-t^2)$

[Sol] $x=2t \quad \cdots ①$
$y=2t-t^2 \quad \cdots ②$
From ① and ②,
$$y=-\frac{1}{4}x^2+x$$
Therefore, the locus is
parabola $y=-\frac{1}{4}x^2+x$.



M62b

2. For all real numbers t, find the locus of midpoint $P(x, y)$ of the line segment connecting two points $(t, 2-t^2)$ and $(3t, 3t^2-4)$.

Sol] $x=\dfrac{t+3t}{2}=2t$ …①

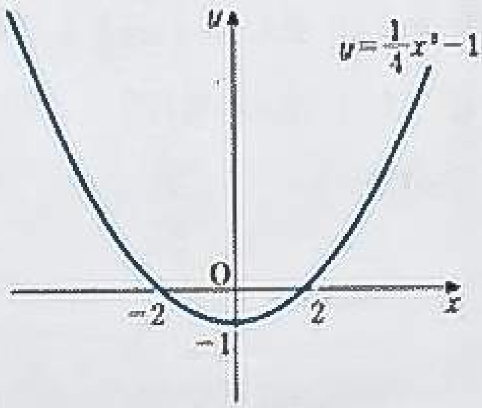
$y=\dfrac{(2-t^2)+(3t^2-4)}{2}=t^2-1$ …②

From ① and ②,

$y=\dfrac{1}{4}x^2-1$

Therefore, the locus is

parabola $y=\dfrac{1}{4}x^2-1$.



3. For $0 \leqslant t \leqslant 2$, find the locus of midpoint $P(x, y)$ of the line segment connecting two points $(3t+2, t^2+1)$ and $(-t-4, t^2-5)$.

Sol] $x=\dfrac{(3t+2)+(-t-4)}{2}=t-1$ …①

$y=\dfrac{(t^2+1)+(t^2-5)}{2}=t^2-2$ …②

From ① and ②,

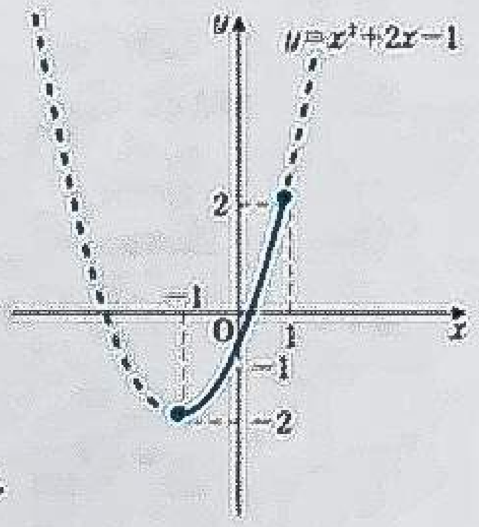
$y=x^2+2x-1$

Since $0 \leqslant t \leqslant 2$, from ①,

$-1 \leqslant x \leqslant 1$

Therefore, the locus is

parabola $y=x^2+2x-1$ for $-1 \leqslant x \leqslant 1$.



Loci 2

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Ex. For all real numbers t, find the locus of vertex P of parabola $y=x^2+2tx+4$.

[Sol] $y=x^2+2tx+4$
$=(x+t)^2-t^2+4$
Let point P be (x, y).
$x=-t$ …①
$y=-t^2+4$ …②
From ① and ②,
$y=-x^2+4$
Therefore, the locus is parabola $y=-x^2+4$.

1. For all real numbers t, find the locus of vertex P of parabola $y=x^2-4tx+1$.

[Sol] $y=x^2-4tx+1$
$=(x-2t)^2-4t^2+1$
Let point P be (x, y).
$x=2t$ …①
$y=-4t^2+1$ …②
From ① and ②,
$y=-x^2+1$
Therefore, the locus is parabola $y=-x^2+1$.

M63b

2. For all real numbers t, find the locus of vertex P of parabola
$y=x^2-2(t+1)x+2t^2-t$.

[Sol] $y=x^2-2(t+1)x+2t^2-t$
$=[x-(t+1)]^2+t^2-3t-1$
Let point P be (x, y).
$x=t+1 \quad \cdots①$
$y=t^2-3t-1 \quad \cdots②$
From ① and ②,
$y=x^2-5x+3$
Therefore, the locus is **parabola $y=x^2-5x+3$.**

3. For all real numbers t, find the locus of center P of circle
$x^2+y^2-4tx+2(t+1)y+5t^2+2t=0$.

[Sol] Rearranging the equation of the circle,
$(x-2t)^2+[y+(t+1)]^2=1$
Let point P be (x, y).
$x=2t \quad \cdots①$
$y=-t-1 \quad \cdots②$
From ① and ②,
$y=-\frac{1}{2}x-1$
Therefore, the locus is **line $y=-\frac{1}{2}x-1$ [$x+2y+2=0$].**

Loci 2

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Ex. For all real numbers t, find the locus of center P of circle $x^2+y^2-4tx+2(t+1)y+6t^2-2t+1=0$ …①.

[Sol] From ①, $(x-2t)^2+[y+(t+1)]^2=-t^2+4t$ …②

Also, since $-t^2+4t>0$, $t(t-4)<0$ ⟵ From the condition $-t^2+4t>0$ for ② to represent a circle

$\therefore\ 0<t<4$ …③

Let point P be (x, y). From ②,

$x=2t$ …④

$y=-t-1$ …⑤

From ④ and ⑤, $y=-\frac{1}{2}x-1$

From ③ and ④, $0<x<8$

Therefore, the locus is

line $y=-\frac{1}{2}x-1$ for $0<x<8$.

1. For all real numbers t, find the locus of center P of circle $x^2+y^2-2(t+1)x+6ty+11t^2+2t-3=0$ …①.

[Sol] From ①, $[x-(t+1)]^2+(y+3t)^2=-t^2+4$ …②

Also, since $-t^2+4>0$, $(t+2)(t-2)<0$

$\therefore\ -2<t<2$ …③

Let point P be (x, y). From ②,

$x=t+1$ …④

$y=-3t$ …⑤

From ④ and ⑤, $y=-3x+3$

From ③ and ④, $-1<x<3$

Therefore, the locus is

line $y=-3x+3$ $[3x+y-3=0]$ for $-1<x<3$.

M64b

2. For all real numbers t, find the locus of center P of circle
$x^2+y^2-4tx+2(3t-1)y+12t^2-8t+1=0$ ···①.

[Sol] From ①, $(x-2t)^2+\{y+(3t-1)\}^2=t^2+2t$ ···②
Also, since $t^2+2t>0$, $t(t+2)>0$
$\therefore\ t<-2,\ 0<t$ ···③
Let point P be (x, y). From ②,
$x=2t$ ···④
$y=-3t+1$ ···⑤
From ④ and ⑤, $y=-\dfrac{3}{2}x+1$
From ③ and ④, $x<-4,\ 0<x$
Therefore, the locus is
line $y=-\dfrac{3}{2}x+1$ $[3x+2y-2=0]$ for $x<-4,\ 0<x$.

3. For all real numbers t, find the locus of center P of circle
$x^2+y^2-2tx+2(t-1)y=0$ ···①.

[Sol] From ①, $(x-t)^2+\{y+(t-1)\}^2=2t^2-2t+1$ ···②
Also, $2t^2-2t+1=2\left(t-\dfrac{1}{2}\right)^2+\dfrac{1}{2}>0$
Therefore, ② represents a circle for all values of t.
Let point P be (x, y). From ②,
$x=t$ ···③
$y=-t+1$ ···④
From ③ and ④, $y=-x+1$
Therefore, the locus is
line $y=-x+1$ $[x+y-1=0]$.

Loci 2

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Ex. For all real numbers m, find the locus of the point of intersection $P(x, y)$ of two lines $mx-y=m$ $\cdots$① and $x+my=-1$ $\cdots$②.

[Sol] From ①, $m(x-1)=y$

(i) When $x\neq1$, $m=\dfrac{y}{x-1}$

Since $x\neq1$, $x-1\neq0$

Substituting into ②,

$$x+\frac{y^2}{x-1}=-1$$

So, $x^2+y^2=1$ $\cdots$③

Given ③, if $x=1$, $y=0$

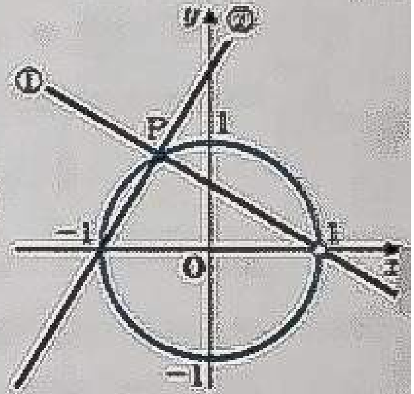
Therefore, when $x\neq1$, point $(1, 0)$ is excluded in ③.

(ii) When $x=1$, from ①, $y=0$

When $x=1$ and $y=0$, ② is not satisfied.

$1+m\cdot0=-1$ is not true.

From (i) and (ii), the locus is a circle with its center at point $(0, 0)$ and radius 1 (excluding point $(1, 0)$).



1. For all real numbers m, find the locus of the point of intersection $P(x, y)$ of two lines $mx+y=2m$ $\cdots$① and $x-my=-2$ $\cdots$②.

[Sol] From ①, $m(x-2)=-y$

(i) When $x\neq2$, $m=-\dfrac{y}{x-2}$

Substituting into ②,

$$x+\frac{y^2}{x-2}=-2$$

So, $x^2+y^2=4$ $\cdots$③

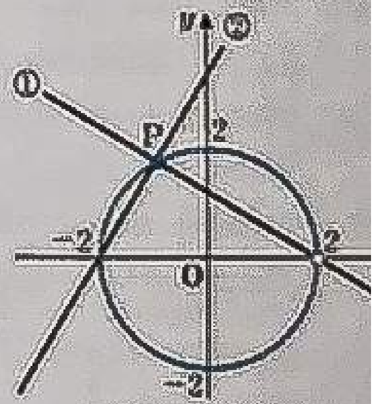
Given ③, if $x=2$, $y=0$

Therefore, when $x\neq2$, point $(2, 0)$ is excluded in ③.

(ii) When $x=2$, from ①, $y=0$

When $x=2$ and $y=0$, ② is not satisfied.

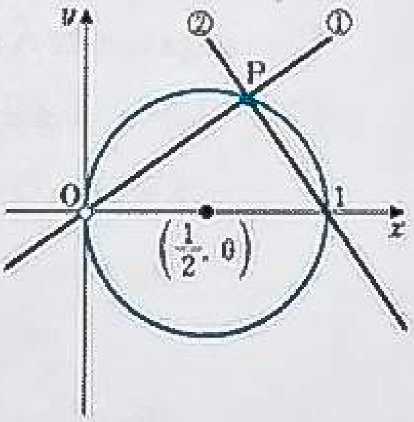
From (i) and (ii), the locus is a circle with its center at point $(0, 0)$ and radius 2 (excluding point $(2, 0)$).



M65b

2. Given two lines $mx-y=0$ $\cdots$① and $x+my=1$ $\cdots$②, solve the following questions.

(1) For all real numbers m, find the locus of the point of intersection $\mathrm{P}(x, y)$ of the two lines.



[Sol] From ①, $mx=y$

(i) When $x \neq 0$, $m=\dfrac{y}{x}$

Substituting into ②,

$$x+\frac{y^2}{x}=1$$

So, $\left(x-\dfrac{1}{2}\right)^2+y^2=\dfrac{1}{4}$ $\cdots$③

Given ③, if $x=0$, $y=0$

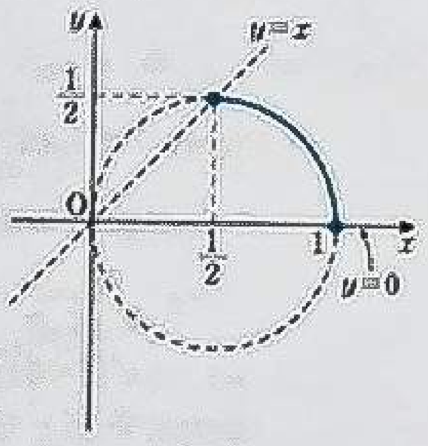
Therefore, when $x \neq 0$, point $(0, 0)$ is excluded in ③.

(ii) When $x=0$, from ①, $y=0$

When $x=0$ and $y=0$, ② is not satisfied.

From (i) and (ii), the locus is a circle with its center at point $\left(\dfrac{1}{2}, 0\right)$ and radius $\dfrac{1}{2}$ (excluding point $(0, 0)$).

(2) Draw the locus of the point of intersection of the two lines when $0 \leqq m \leqq 1$.



[Sol] From ①, $y=mx$ $\cdots$④

Line ④ is the line with slope m passing through point $(0, 0)$.

When $m=1$,

from ④, $y=x$ $\cdots$⑤

The point of intersection of the locus found in question (1) and ⑤ is $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$.

When $m=0$,

from ④, $y=0$ $\cdots$⑥

The point of intersection of the locus found in question (1) and ⑥ is $(1, 0)$.

Therefore, when $0 \leqq m \leqq 1$,

the locus is as shown in the diagram above.

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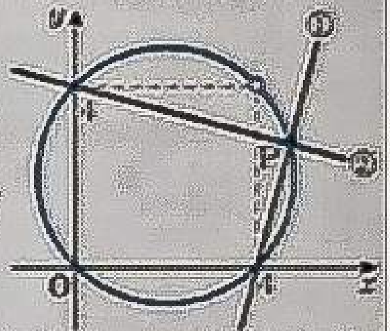
EX For all real numbers m, find the locus of the point of intersection $P(x, y)$ of two lines $mx+y=4m$ ···① and $x-my=-4m$ ···②.



[Sol] From ①, $m(x-4)=-y$

(i) When $x \neq 4$, $m=-\dfrac{y}{x-4}$

Substituting into ②, $x+\dfrac{y^2}{x-4}=\dfrac{4y}{x-4}$

So, $(x-2)^2+(y-2)^2=8$ ···③

Given ③, if $x=4$, $y=0, 4$

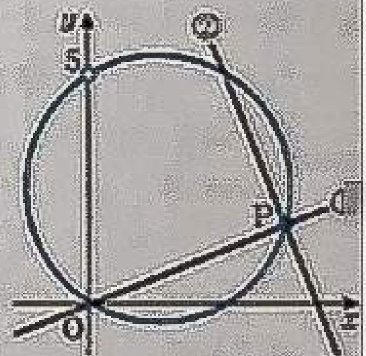
Therefore, when $x \neq 4$, points $(4, 0)$ and $(4, 4)$ are excluded in ③.

(ii) When $x=4$, from ①, $y=0$

Substituting $x=4$ and $y=0$ into ②, when $m=-1$, ② is satisfied. So, point $(4, 0)$ satisfies the condition.

From (i) and (ii), the locus is a circle with its center at point $(2, 2)$ and radius $2\sqrt{2}$ (excluding point $(4, 4)$).

1. For all real numbers m, find the locus of the point of intersection $P(x, y)$ of two lines $mx-y=0$ ···① and $x+my=5m+3$ ···②.



[Sol] From ①, $mx=y$

(i) When $x \neq 0$, $m=\dfrac{y}{x}$

Substituting into ②, $x+\dfrac{y^2}{x}=\dfrac{5y}{x}+3$

So, $\left(x-\dfrac{3}{2}\right)^2+\left(y-\dfrac{5}{2}\right)^2=\dfrac{17}{2}$ ···③

Given ③, if $x=0$, $y=0, 5$

Therefore, when $x \neq 0$, points $(0, 0)$ and $(0, 5)$ are excluded in ③.

(ii) When $x=0$, from ①, $y=0$

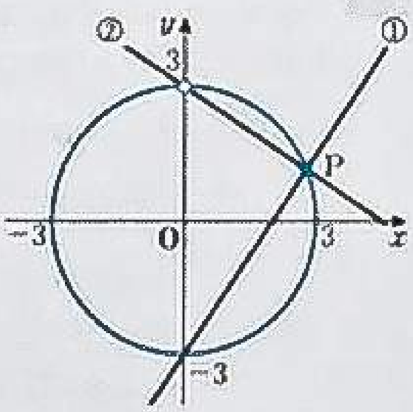
Substituting $x=0$ and $y=0$ into ②, when $m=-\dfrac{3}{5}$, ② is satisfied. So, point $(0, 0)$ satisfies the condition.

From (i) and (ii), the locus is a circle with its center at point $\left(\dfrac{3}{2}, \dfrac{5}{2}\right)$ and radius $\dfrac{\sqrt{34}}{2}$ (excluding point $(0, 5)$).

M66b

2. Given two lines $mx-y=3$ ⋯① and $x+my=3m$ ⋯②, solve the following questions.

(1) For all real numbers m, find the locus of the point of intersection $\mathrm{P}(x, y)$ of the two lines.



[Sol] From ①, $mx=y+3$

(i) When $x \neq 0$, $m=\dfrac{y+3}{x}$

Substituting into ②,

$$x+\frac{y(y+3)}{x}=\frac{3(y+3)}{x}$$

So, $x^2+y^2=9$ ⋯③

Given ③, if $x=0$, $y=\pm 3$

Therefore, when $x \neq 0$, points $(0, 3)$ and $(0, -3)$ are excluded in ③.

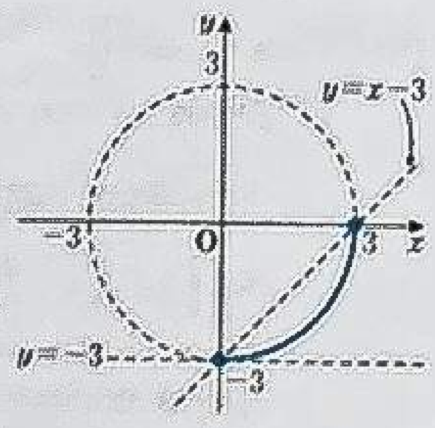
(ii) When $x=0$, from ①, $y=-3$

Substituting $x=0$ and $y=-3$ into ②, when $m=0$, ② is satisfied.

So, point $(0, -3)$ satisfies the condition.

From (i) and (ii), the locus is a **circle with its center at point $(0, 0)$ and radius 3 (excluding point $(0, 3)$).**

(2) Draw the locus of the point of intersection of the two lines when $0 \leqq m \leqq 1$.



[Sol] From ①, $y=mx-3$ ⋯④

Line ④ is the line with slope m passing through point $(0, -3)$.

When $m=1$,

from ④, $y=x-3$ ⋯⑤

The points of intersection of the locus found in question (1) and ⑤ are $(3, 0)$ and $(0, -3)$.

When $m=0$,

from ④, $y=-3$ ⋯⑥

The point of intersection of the locus found in question (1) and ⑥ is $(0, -3)$.

Therefore, when $0 \leqq m \leqq 1$,

the locus is as shown in the diagram above.

M67a

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Loci 2

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Ex. Find the equations of the bisectors of the angles formed by two lines $x-2y-2=0$ and $4x-2y+1=0$.

[Sol] Let $P(x, y)$ be a point on the bisectors. Since point P is equidistant from each given line,

$$\frac{|x-2y-2|}{\sqrt{1^2+(-2)^2}}=\frac{|4x-2y+1|}{\sqrt{4^2+(-2)^2}}$$

Distance from a Point to a Line (M25)

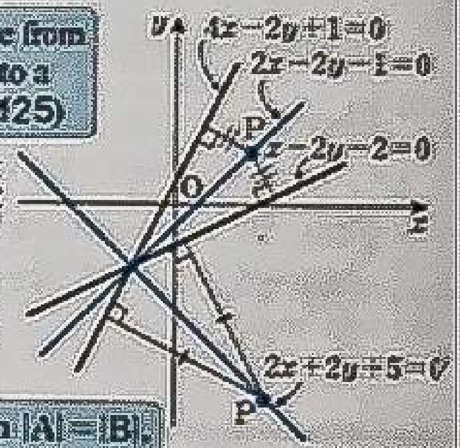
Therefore,

$$2(x-2y-2)=\pm(4x-2y+1)$$

Thus, the equations are

$$2x+2y+5=0,\ 2x-2y-1=0$$

When $|A|=|B|$, $A=\pm B$



1. Find the equations of the bisectors of the angles formed by two lines $3x+2y-5=0$ and $2x-3y+4=0$.

[Sol] Let $P(x, y)$ be a point on the bisectors.

Since point P is equidistant from each given line,

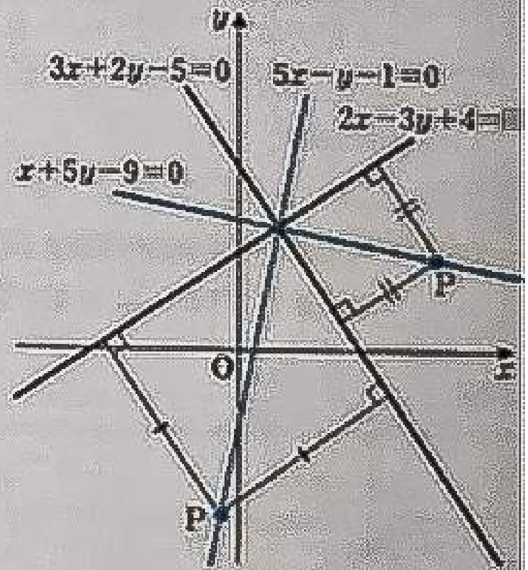
$$\frac{|3x+2y-5|}{\sqrt{3^2+2^2}}=\frac{|2x-3y+4|}{\sqrt{2^2+(-3)^2}}$$

Therefore,

$$3x+2y-5=\pm(2x-3y+4)$$

Thus, the equations are

$$x+5y-9=0,\ 5x-y-1=0$$



M67b

2. Find the equations of the bisectors of the angles formed by two lines $4x+3y-8=0$ and $5y+3=0$.

[Sol] Let $P(x, y)$ be a point on the bisectors. Since point P is equidistant from each given line,

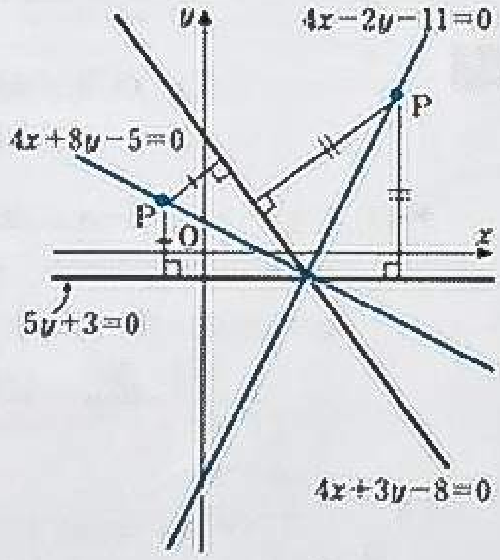
$$\frac{|4x+3y-8|}{\sqrt{4^2+3^2}}=\frac{|0\cdot x+5y+3|}{\sqrt{0^2+5^2}}$$

Therefore,

$$4x+3y-8=\pm(5y+3)$$

Thus, the equations are

$$4x-2y-11=0,\ 4x+8y-5=0$$



3. The ratio of the distances from one point to lines $x+y-1=0$ and $x-y-2=0$ is $2:1$. Find the equations of the locus of this point.

[Sol] Let $P(x, y)$ be a point on the locus. Since the ratio of the distances to lines $x+y-1=0$ and $x-y-2=0$ is $2:1$,

$$\frac{|x+y-1|}{\sqrt{1^2+1^2}}:\frac{|x-y-2|}{\sqrt{1^2+(-1)^2}}=2:1$$

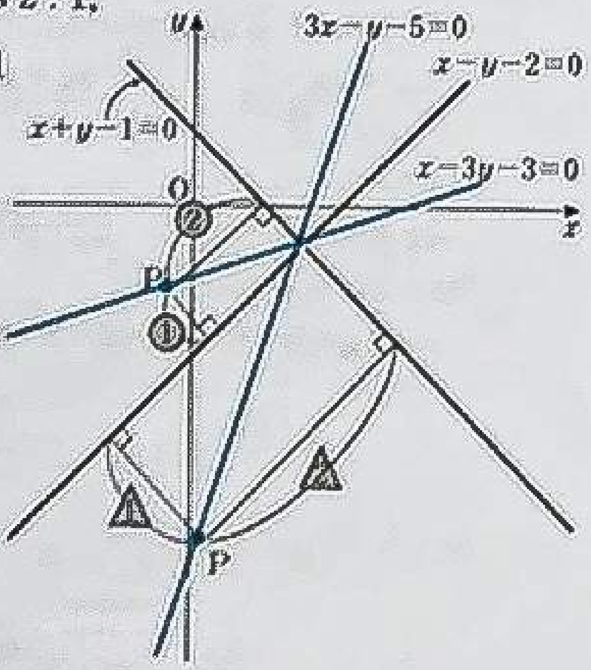
$$2|x-y-2|=|x+y-1|$$

Therefore,

$$2(x-y-2)=\pm(x+y-1)$$

Thus, the equations are

$$x-3y-3=0,\ 3x-y-5=0$$



Loci 2

Name

Date / /

Time : to :

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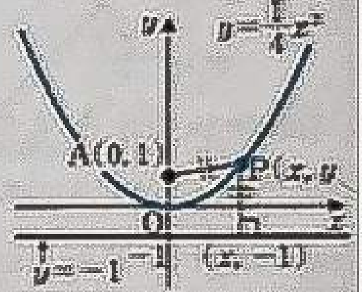
EX Given point A(0, 1) and line $y=-1$, find the locus of point P which is equidistant from point A and the line.

[Sol] Let point P be (x, y). AP=(distance from $y=-1$ to point P)

$$\sqrt{x^2+(y-1)^2}=|y+1|$$
$$x^2+(y-1)^2=(y+1)^2$$
$$\therefore\ y=\frac{1}{4}x^2$$

Therefore, the locus is parabola $y=\frac{1}{4}x^2$.

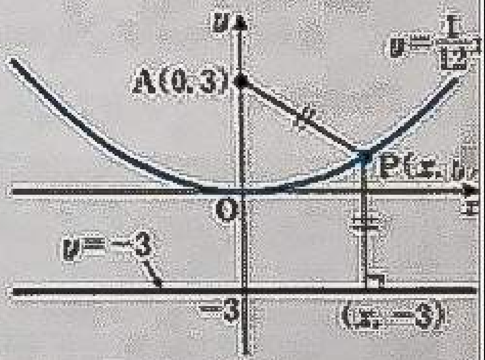


1. Given point A(0, 3) and line $y=-3$, find the locus of point P which equidistant from point A and the line.

[Sol] Let point P be (x, y).

$$\sqrt{x^2+(y-3)^2}=|y+3|$$
$$x^2+(y-3)^2=(y+3)^2$$
$$\therefore\ y=\frac{1}{12}x^2$$

Therefore, the locus is **parabola $y=\frac{1}{12}x^2$.**

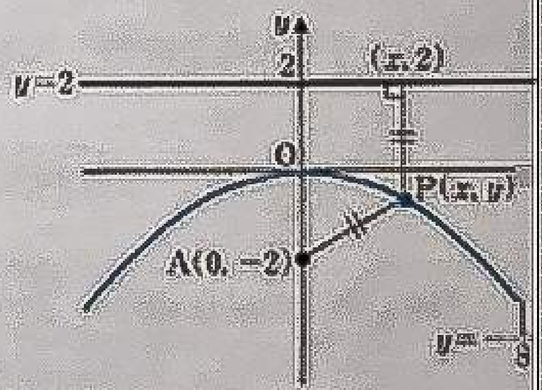


2. Given point A(0, -2) and line $y=2$, find the locus of point P which equidistant from point A and the line.

[Sol] Let point P be (x, y).

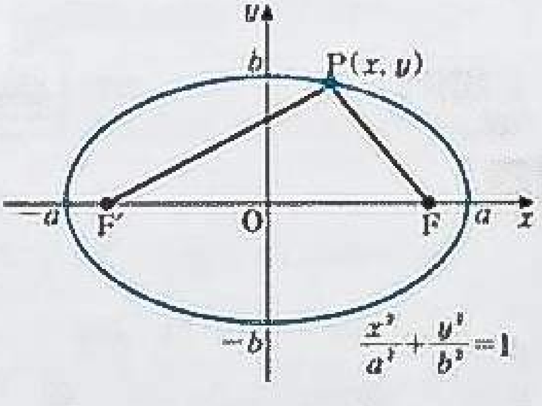
$$\sqrt{x^2+(y+2)^2}=|y-2|$$
$$x^2+(y+2)^2=(y-2)^2$$
$$\therefore\ y=-\frac{1}{8}x^2$$

Therefore, the locus is **parabola $y=-\frac{1}{8}x^2$.**



M68b

The locus of point P, for which the sum of the distances from two fixed points F and F′ on a plane is constant, is called an ***ellipse***.

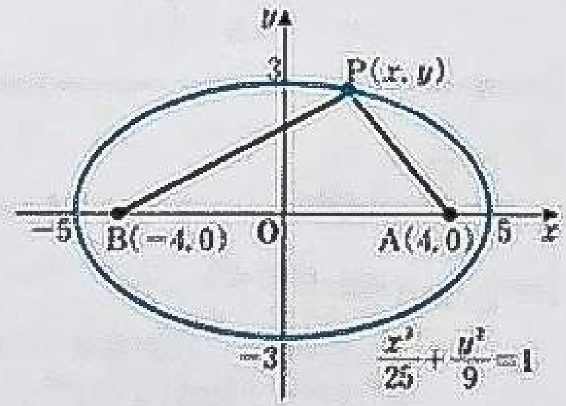


Generally, an ellipse is represented as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \ (a>0, b>0)$$

3. Given points A(4, 0) and B(−4, 0) where AP+BP=10, find the locus of point P.

[Sol] Let point P be (x, y).



$$AP=\sqrt{(x-4)^2+y^2}$$

$$BP=\sqrt{(x+4)^2+y^2}$$

Therefore,

$$\sqrt{(x-4)^2+y^2}+\sqrt{(x+4)^2+y^2}=10$$

$$\sqrt{(x-4)^2+y^2}=10-\sqrt{(x+4)^2+y^2}$$

Squaring both sides

$$(x-4)^2+y^2=100-20\sqrt{(x+4)^2+y^2}+(x+4)^2+y^2$$

Simplifying

$$5\sqrt{(x+4)^2+y^2}=4x+25$$

Squaring both sides

$$25[(x+4)^2+y^2]=(4x+25)^2$$

So, $\frac{x^2}{25}+\frac{y^2}{9}=1$

Thus, the locus is ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$.

Loci 2

Name

Date / /

Time : to :

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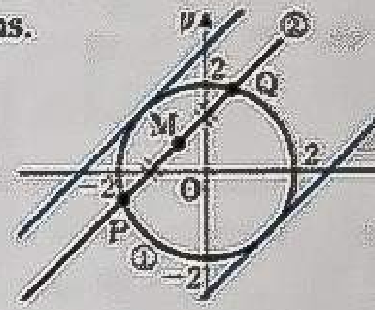
1\. Given that circle $x^2+y^2=4$ ⋯① and line $y=x+k$ ⋯② intersect at two different points P and Q, solve the following questions.



(1) Find the range of the values of constant k.

[Sol] From ① and ②,

$x^2+(x+k)^2=4$ ← Substituting ② into ①

$\therefore\ 2x^2+2kx+k^2-4=0$ ⋯③

Since ① and ② intersect at two different points,

from ③, $\dfrac{D}{4}=k^2-2(k^2-4)$

$=-k^2+8>0$

$\therefore\ k^2<8$

$\therefore\ -2\sqrt{2}<k<2\sqrt{2}$

Alternative Solution

Since the distance d from center $(0,0)$ to line $y=x+k$, i.e. $x-y+k=0$ is smaller than 2,

$d=\dfrac{|0-0+k|}{\sqrt{1^2+(-1)^2}}=\dfrac{|k|}{\sqrt{2}}<2$

$\therefore\ -2\sqrt{2}<k<2\sqrt{2}$

(2) Express the midpoint $M(x,y)$ of line segment PQ in terms of k.

[Sol] Let the x-coordinates of points P and Q be α and β.

From ③, $\alpha+\beta=-\dfrac{2k}{2}=-k$ ← Since α and β are the real solutions of $2x^2+2kx+k^2-4=0$, $\alpha+\beta=-\dfrac{2k}{2}$ (From Root-Coefficient Relationships (J131))

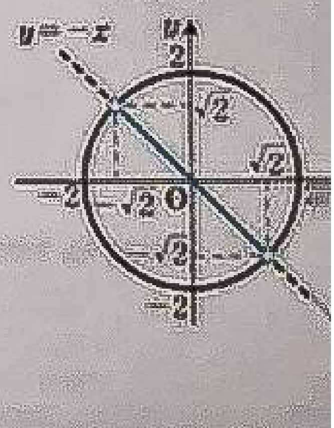
$\therefore\ x=\dfrac{\alpha+\beta}{2}=-\dfrac{k}{2}$ ⋯④

Since M lies on line ②,

$y=-\dfrac{k}{2}+k=\dfrac{k}{2}$ ⋯⑤

$\therefore\ M\left(-\dfrac{k}{2},\dfrac{k}{2}\right)$

(3) When the value of k changes in the range found in question (1), find the locus of midpoint M.



[Sol] From ④ and ⑤, $y=-x$ ← Eliminating k

However, from (1) and ④,

$-\sqrt{2}<x<\sqrt{2}$ ← Since $-2\sqrt{2}<k<2\sqrt{2}$, $-\sqrt{2}<-\dfrac{k}{2}<\sqrt{2}$

Therefore, the locus of midpoint M is

line $y=-x$ for $-\sqrt{2}<x<\sqrt{2}$.

M69b

2. For real numbers x and y that satisfy the relationship $x^2+y^2=1$, find the locus of point $P(x+y, xy)$ and draw the locus.

[Sol] Let $x+y=X$ and $xy=Y$.

Since $x^2+y^2=1$, $(x+y)^2-2xy=1$.

$\therefore X^2-2Y=1$

$\therefore Y=\frac{1}{2}X^2-\frac{1}{2}$ $\cdots$①

Also, x and y are the real solutions of equation $t^2-Xt+Y=0$.

$\therefore D=X^2-4Y \geqslant 0$ $\cdots$②

When $x+y=X$ and $xy=Y$, then x and y are the real solutions of quadratic equation $t^2-Xt+Y=0$. (From Root-Coefficient Relationships (J 147))

From ① and ②, $-X^2+2 \geqslant 0$

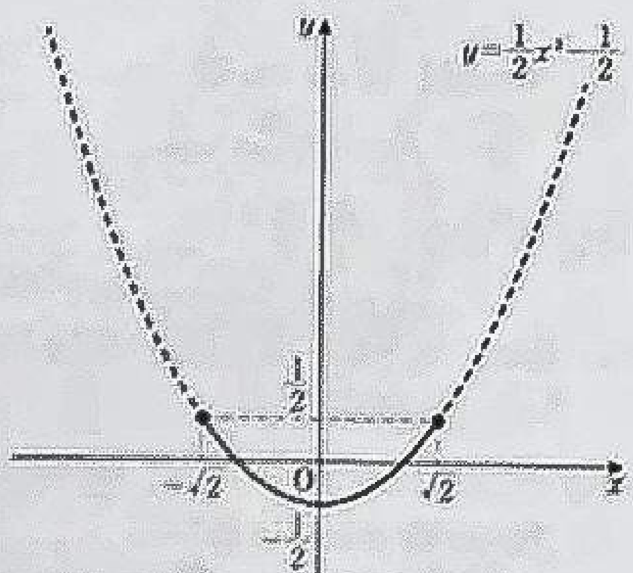
$\therefore -\sqrt{2} \leqslant X \leqslant \sqrt{2}$ $\cdots$③

Replacing variables X and Y of ① and ③ with x and y,

$y=\frac{1}{2}x^2-\frac{1}{2}$, $-\sqrt{2} \leqslant x \leqslant \sqrt{2}$

Therefore, the locus is

parabola $y=\frac{1}{2}x^2-\frac{1}{2}$ for $-\sqrt{2} \leqslant x \leqslant \sqrt{2}$ as shown below.



M70a

Loci 2

Name

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Time : to :

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1. For all real numbers t, find the locus of vertex P of parabola $y=-x^2+4tx-2t^2+1$. ➡ Me

[Sol] $y=-x^2+4tx-2t^2+1$
$=-(x-2t)^2+2t^2+1$

Let point P be (x, y).

$x=2t \quad \cdots ①$

$y=2t^2+1 \quad \cdots ②$

From ① and ②,

$y=\frac{1}{2}x^2+1$

Therefore, the locus is parabola $y=\frac{1}{2}x^2+1$.

2. For all real numbers m, find the locus of the point of intersection P(x, y) of two lines $mx-y=-2m$ $\cdots ①$ and $x+my=2$ $\cdots ②$. ➡ M6

[Sol] From ①, $m(x+2)=y$

(i) When $x\neq-2$, $m=\frac{y}{x+2}$

Substituting into ②,

$x+\frac{y^2}{x+2}=2$

So, $x^2+y^2=4 \quad \cdots ③$

Given ③, if $x=-2$, $y=0$

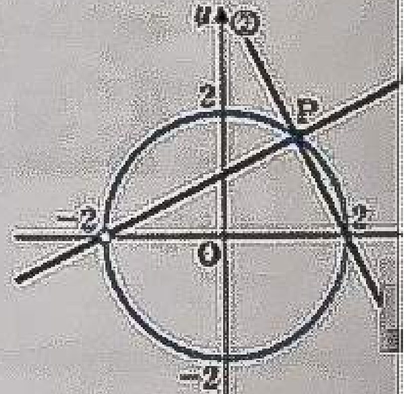
Therefore, when $x\neq-2$,

point $(-2, 0)$ is excluded in ③.

(ii) When $x=-2$, from ①, $y=0$

When $x=-2$ and $y=0$, ② is not satisfied.

From (i) and (ii), the locus is a circle with its center at point $(0, 0)$ and radius 2 (excluding point $(-2, 0)$).



M70b

3. Find the equations of the bisectors of the angles formed by two lines $2x+y-3=0$ and $x-2y+1=0$. ➡ M67

[Sol] Let $P(x, y)$ be a point on the bisectors.
Since point P is equidistant from each given line,

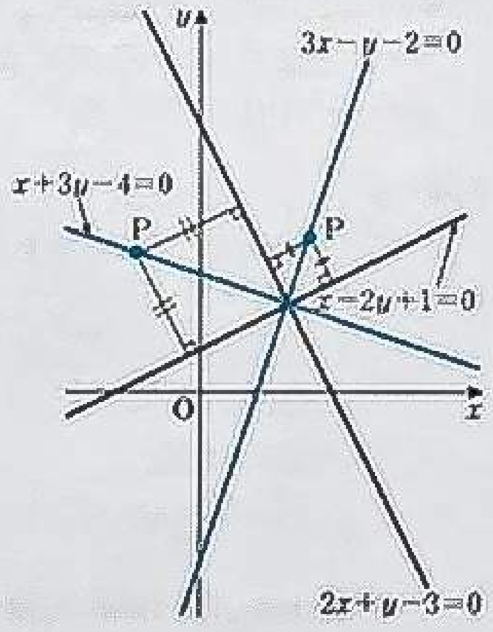
$$\frac{|2x+y-3|}{\sqrt{2^2+1^2}}=\frac{|x-2y+1|}{\sqrt{1^2+(-2)^2}}$$

Therefore,

$$2x+y-3=\pm(x-2y+1)$$

Thus, the equations are

$$x+3y-4=0,\ 3x-y-2=0$$



4. Given point $A(0, 4)$ and line $y=-4$, find the locus of point P which is equidistant from point A and the line. ➡ M68

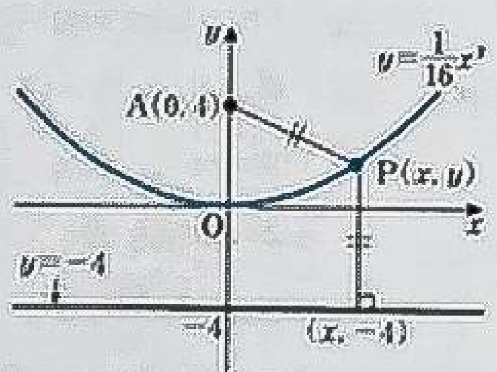
[Sol] Let point P be (x, y).

$$\sqrt{x^2+(y-4)^2}=|y+4|$$

$$x^2+(y-4)^2=(y+4)^2$$

$$\therefore\ y=\frac{1}{16}x^2$$

Therefore, the locus is parabola $y=\frac{1}{16}x^2$.



Regions

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Consider a group of points satisfying the inequality $y \geqq x+1 \cdots$ ①.

Let (x, y) be the points satisfying ①.

When $x=-2$, $y \geqq -2+1=-1$

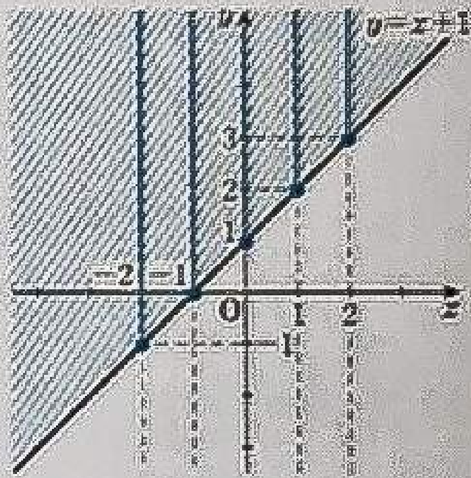
When $x=-1$, $y \geqq -1+1=0$

When $x=0$, $\quad y \geqq 0+1=1$

When $x=1$, $\quad y \geqq 1+1=2$

When $x=2$, $\quad y \geqq 2+1=3$

$\vdots \qquad \vdots$

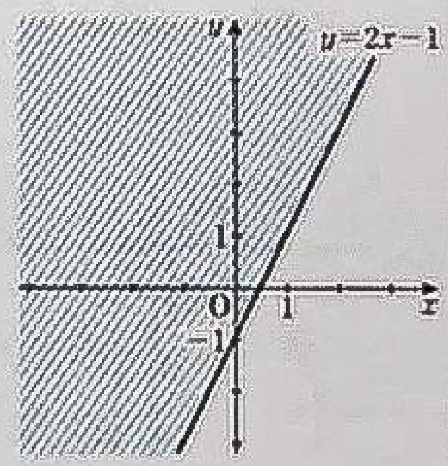


From the above, points (x, y) satisfying ① are above or on the line $y=x+1$. The group of points (x, y) satisfying the inequality of x and y is called a ***region***.

Shade the region satisfying each given inequality.

Ex. $y>2x-1$

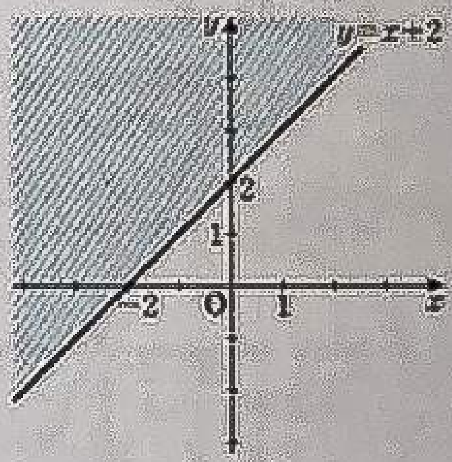
[Sol]



The boundary is not included.

(1) $y>x+2$

[Sol]

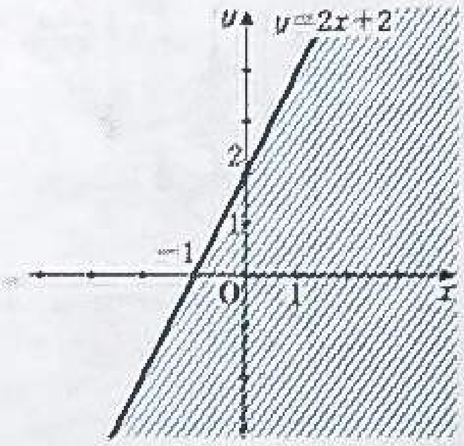


The boundary is not included.

(2) $2x - y > -2$

[Sol] Rearranging,

$$y < 2x + 2$$

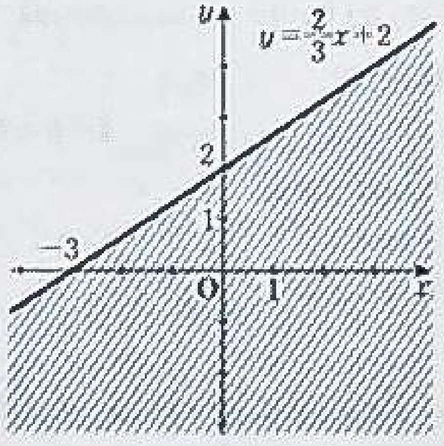


The boundary is not included.

(4) $2x - 3y > -6$

[Sol] Rearranging,

$$y < \frac{2}{3}x + 2$$

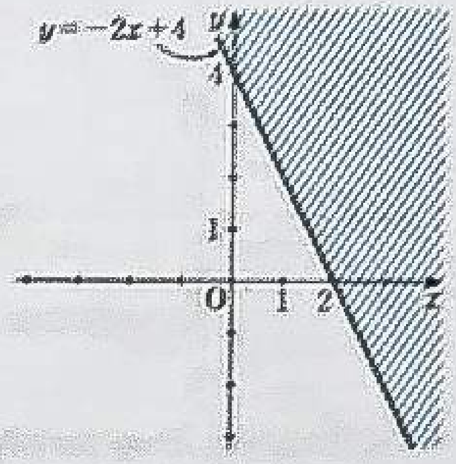


The boundary is not included.

(3) $2x + y - 4 \geqslant 0$

[Sol] Rearranging,

$$y \geqslant -2x + 4$$

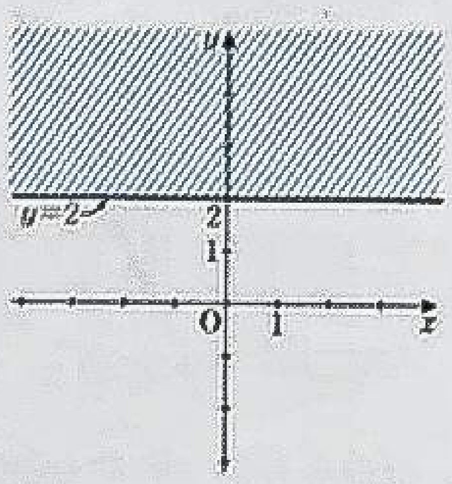


The boundary is included.

(5) $3y - 6 \geqslant 0$

[Sol] Rearranging,

$$y \geqslant 2$$



The boundary is included.

M72a

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Regions

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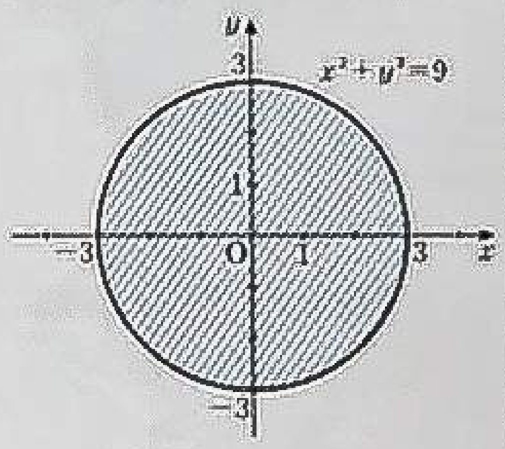
Shade the region satisfying each given inequality.

Ex. $x^2+y^2 \leqq 9$

[Sol]

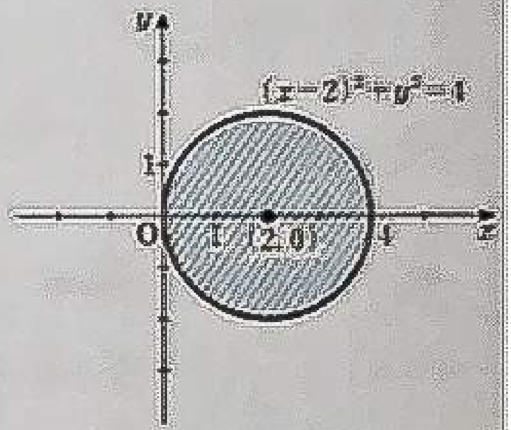


The boundary is included.

(2) $x^2+y^2<4x$

[Sol] Rearranging,

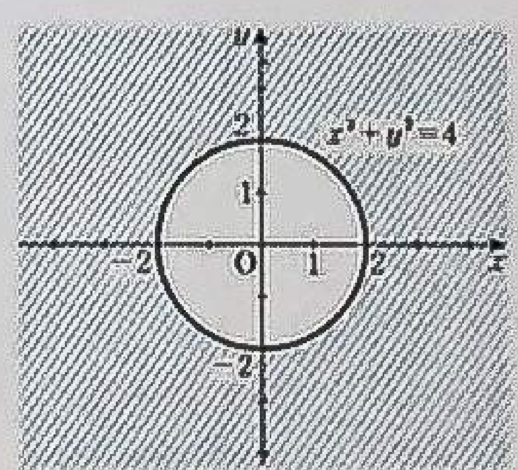
$$(x-2)^2+y^2<4$$



The boundary is not included.

(1) $x^2+y^2 \geqq 4$

[Sol]

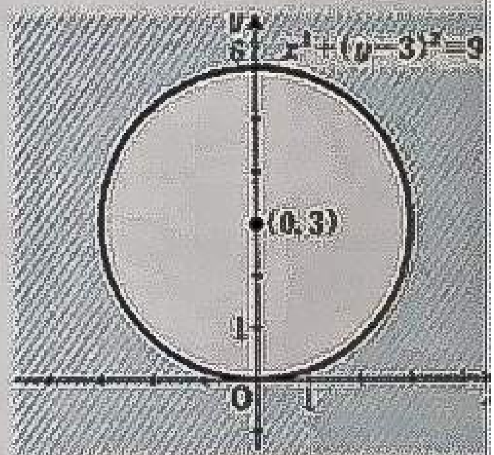


The boundary is included.

(3) $x^2+y^2>6y$

[Sol] Rearranging,

$$x^2+(y-3)^2>9$$



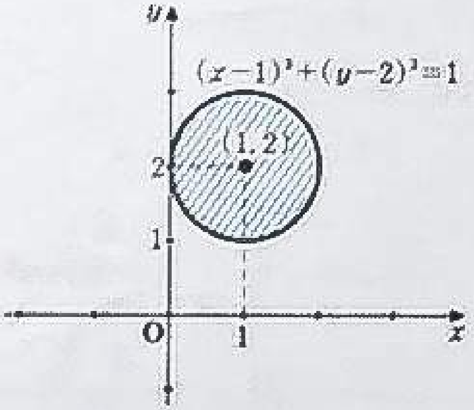
The boundary is not included.

M72b

(4) $x^2+y^2-2x-4y+4\leqq 0$

[Sol] Rearranging,

$$(x-1)^2+(y-2)^2\leqq 1$$

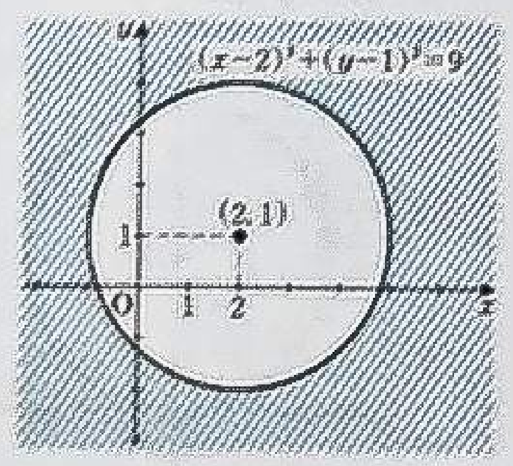


The boundary is included.

(5) $x^2-4x-4>-y^2+2y$

[Sol] Rearranging,

$$(x-2)^2+(y-1)^2>9$$

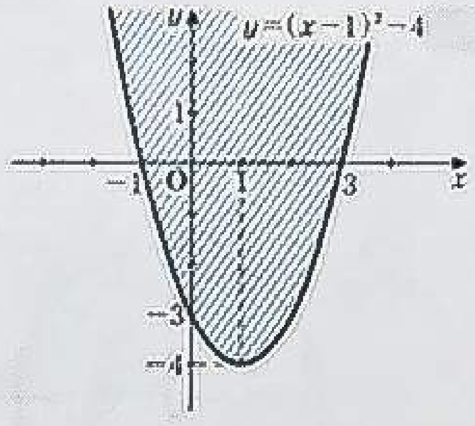


The boundary is not included.

(6) $y\geqq x^2-2x-3$

[Sol] Rearranging,

$$y\geqq (x-1)^2-4$$

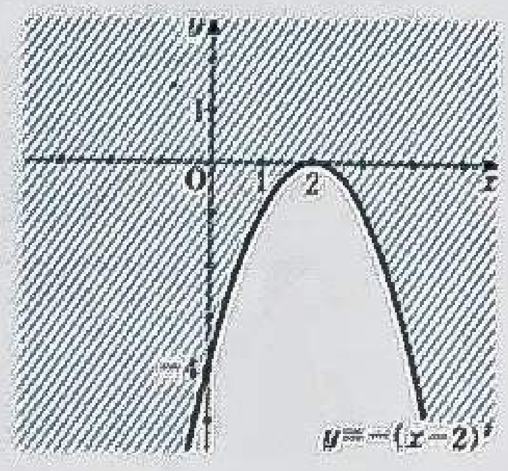


The boundary is included.

(7) $x^2+y>4x-4$

[Sol] Rearranging,

$$y>-(x-2)^2$$



The boundary is not included.

Regions

Name

Date / /

Time : to :

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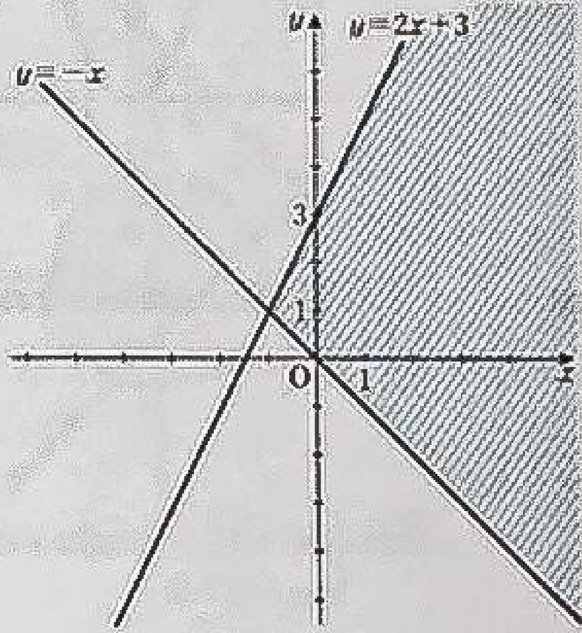
Shade the region satisfying each given system of inequalities.

Ex. $\begin{cases} y < 2x+3 \\ y > -x \end{cases}$

[Sol]



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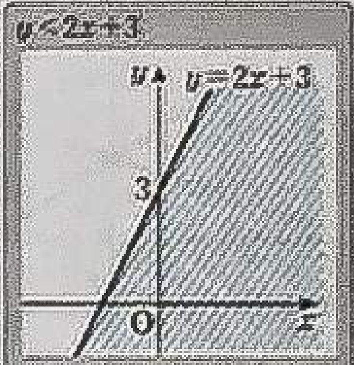

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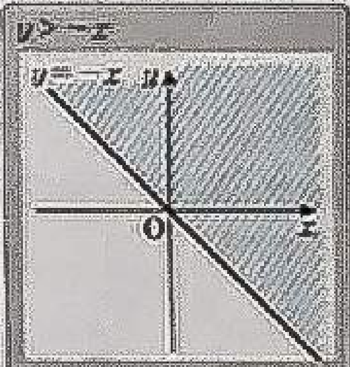
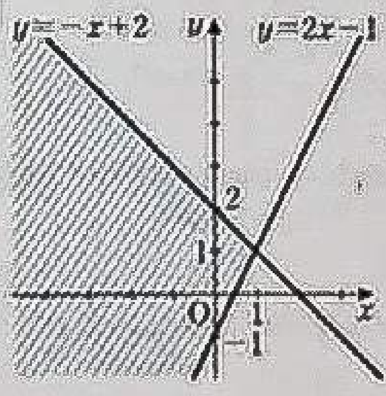


The boundary is not included.

(1) $\begin{cases} y < -x+2 \\ y > 2x-1 \end{cases}$

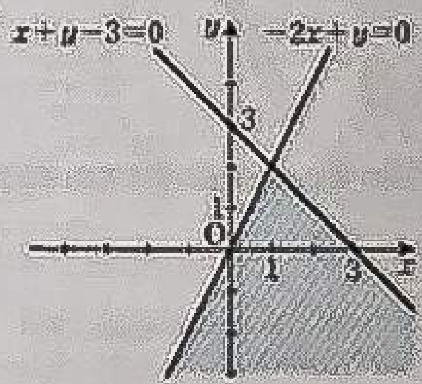
[Sol]



The boundaries are not included.

(2) $\begin{cases} x+y-3 \leqq 0 \\ -2x+y \leqq 0 \end{cases}$

[Sol]

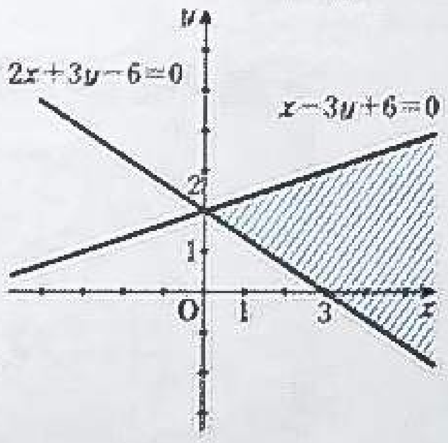


The boundaries are included.

M73b

(3) $\begin{cases} x-3y+6 \geqq 0 \\ 2x+3y-6 \geqq 0 \end{cases}$

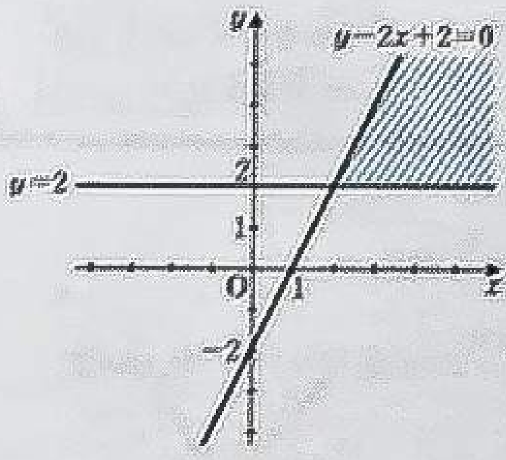
[Sol]



The boundaries are included.

(4) $\begin{cases} y-2x+2<0 \\ y>2 \end{cases}$

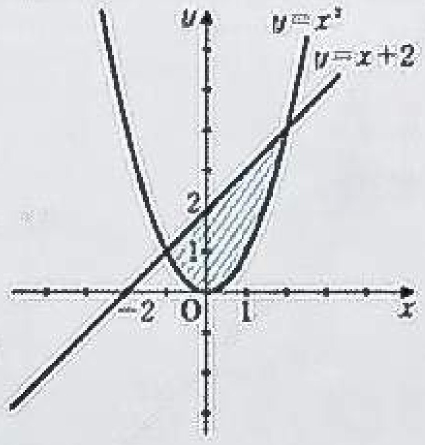
[Sol]



The boundaries are not included.

(5) $\begin{cases} y \geqq x^2 \\ y \leqq x+2 \end{cases}$

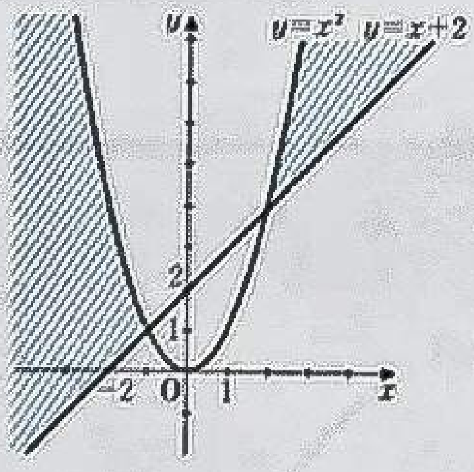
[Sol]



The boundaries are included.

(6) $\begin{cases} y<x^2 \\ y>x+2 \end{cases}$

[Sol]



The boundaries are not included.

Regions

Name

Date / /

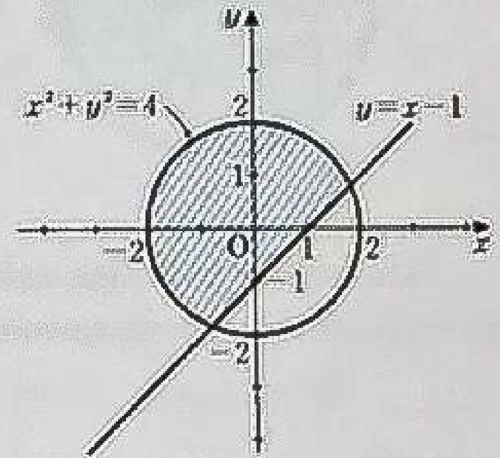
Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	2	3

Shade the region satisfying each given system of inequalities.

(1) $\begin{cases} x^2+y^2<4 \\ y>x-1 \end{cases}$

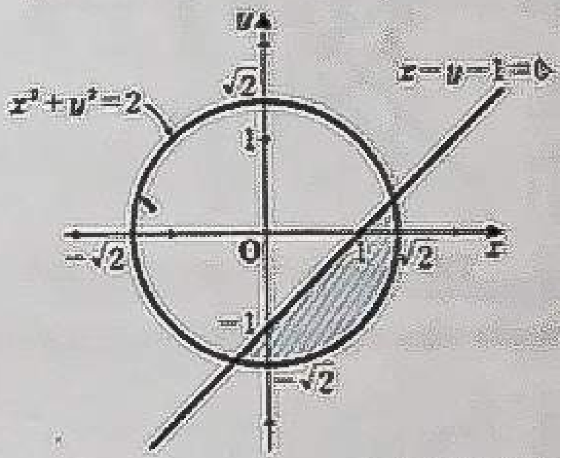
[Sol]



The boundaries are not included.

(3) $\begin{cases} x^2+y^2\leqq 2 \\ x-y-1\geqq 0 \end{cases}$

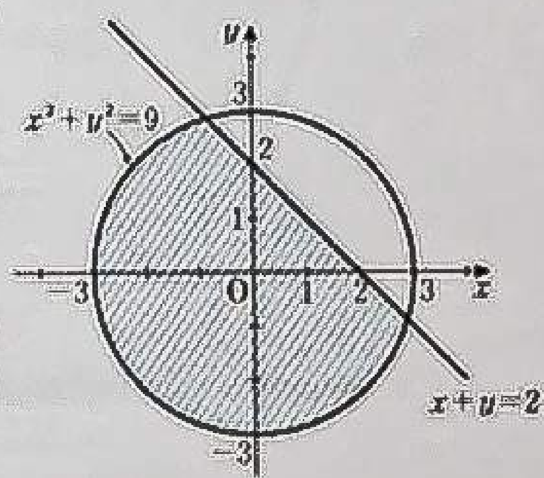
[Sol]



The boundaries are included.

(2) $\begin{cases} x^2+y^2\leqq 9 \\ x+y\leqq 2 \end{cases}$

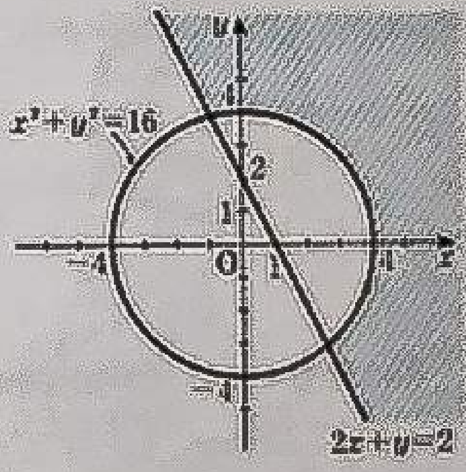
[Sol]



The boundaries are included.

(4) $\begin{cases} x^2+y^2>16 \\ 2x+y>2 \end{cases}$

[Sol]

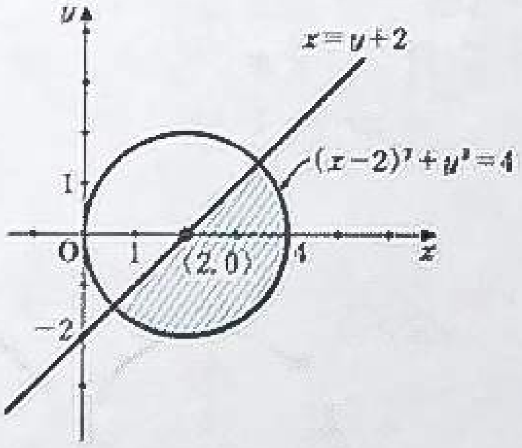


The boundaries are not includ

M74b

(5) $\begin{cases} (x-2)^2+y^2 \leqq 4 \\ x \geqq y+2 \end{cases}$

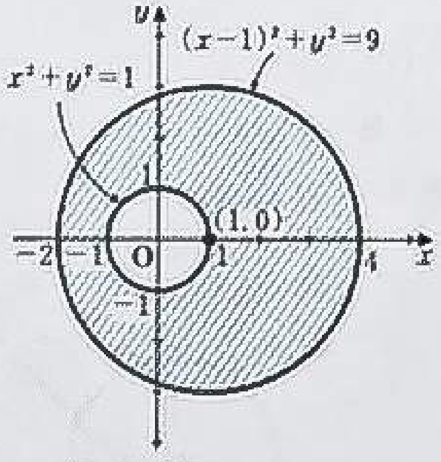
[Sol]



The boundaries are included.

(7) $\begin{cases} x^2+y^2 > 1 \\ (x-1)^2+y^2 < 9 \end{cases}$

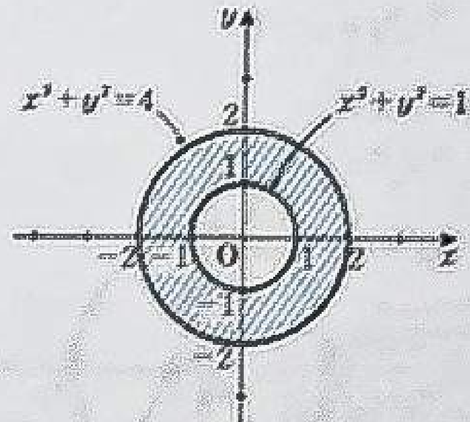
[Sol]



The boundaries are not included.

(6) $\begin{cases} x^2+y^2 \leqq 4 \\ x^2+y^2 \geqq 1 \end{cases}$

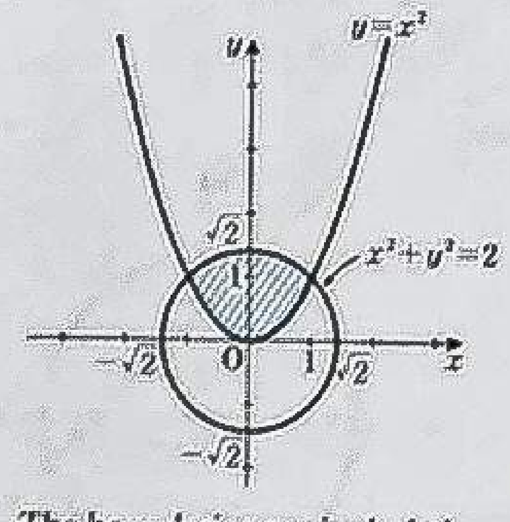
[Sol]



The boundaries are included.

(8) $\begin{cases} x^2+y^2 \leqq 2 \\ y \geqq x^2 \end{cases}$

[Sol]



The boundaries are included.

Regions

Name	
Date	/ /
Time	: to :

100%	~90%	~80%	~70%	69%~
0	—	1	—	2

Shade the region satisfying each given inequality.

Ex. $(x+y)(x-2y)<0$

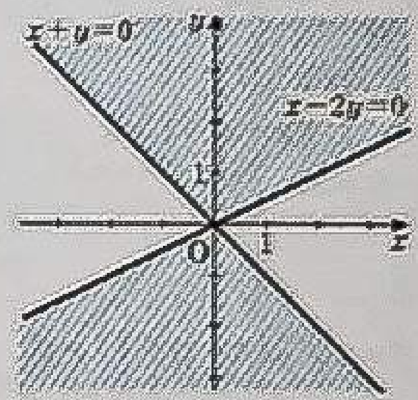
[Sol] When the given inequality is true, either

$$\begin{cases} x+y>0 \\ x-2y<0 \end{cases}$$

or

$$\begin{cases} x+y<0 \\ x-2y>0 \end{cases}$$

is true.



The boundaries are not included.

(1) $(x-y)(x+y-2)\leqq 0$

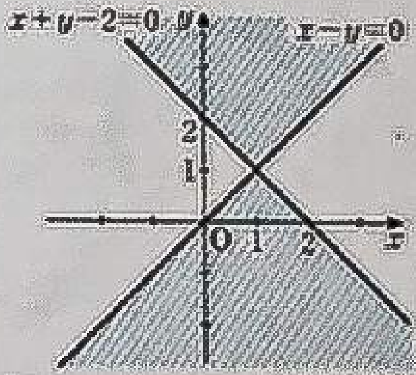
[Sol] When the given inequality is true, either

$$\begin{cases} x-y\geqq 0 \\ x+y-2\leqq 0 \end{cases}$$

or

$$\begin{cases} x-y\leqq 0 \\ x+y-2\geqq 0 \end{cases}$$

is true.



The boundaries are included.

(2) $(x-y-2)(2x+y-4)>0$

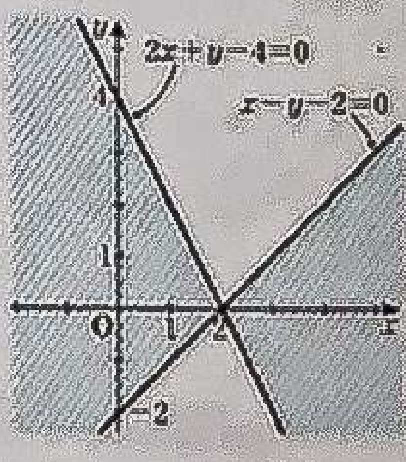
[Sol] When the given inequality is true, either

$$\begin{cases} x-y-2>0 \\ 2x+y-4>0 \end{cases}$$

or

$$\begin{cases} x-y-2<0 \\ 2x+y-4<0 \end{cases}$$

is true.



The boundaries are not include

M74b

(5) $\begin{cases} (x\text{-} \\ x \geqq \end{cases}$

[Sol]

(6)

[Sol

M75b

(3) $y(2x+y-4)<0$

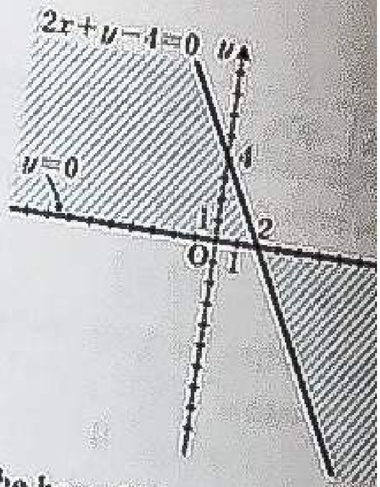
[Sol] When the given inequality is true,
either

$$\begin{cases} y>0 \\ 2x+y-4<0 \end{cases}$$

or

$$\begin{cases} y<0 \\ 2x+y-4>0 \end{cases}$$

is true.



The boundaries are not inclu

(4) $(y-x^2)(x-y+1)\geqq 0$

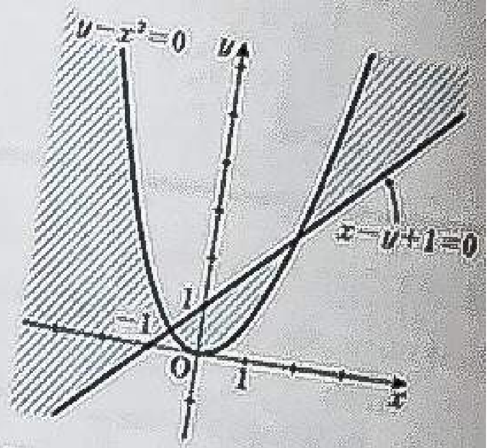
[Sol] When the given inequality is true,
either

$$\begin{cases} y-x^2\geqq 0 \\ x-y+1\geqq 0 \end{cases}$$

or

$$\begin{cases} y-x^2\leqq 0 \\ x-y+1\leqq 0 \end{cases}$$

is true.



The boundaries are included.

(5) $(x^2+y^2-9)(y-x-1)<0$

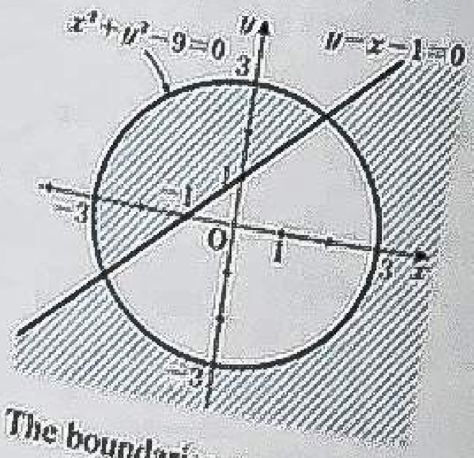
[Sol] When the given inequality is true,
either

$$\begin{cases} x^2+y^2-9>0 \\ y-x-1<0 \end{cases}$$

or

$$\begin{cases} x^2+y^2-9<0 \\ y-x-1>0 \end{cases}$$

is true.



The boundaries are not included.

Regions

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0	—	1	—	2

Shade the region satisfying each given inequality.

(1) $y \geqq |x+1|$

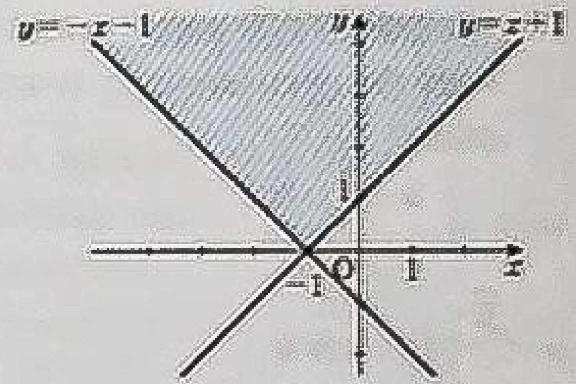
[Sol] $x+1 \geqq 0$ when $x \geqq -1$

$\therefore\ y \geqq x+1$

$x+1 < 0$ when $x < -1$

$\therefore\ y \geqq -(x+1)$

$= -x-1$



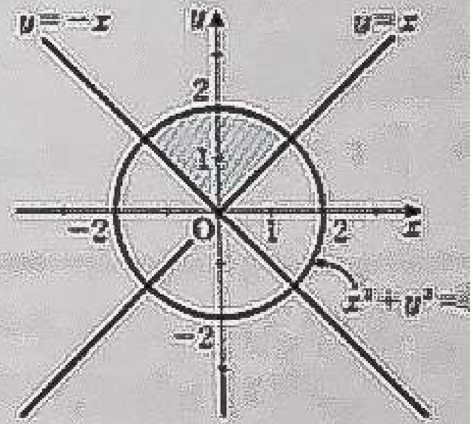
The boundaries are included.

(2) $\begin{cases} y > |x| \\ x^2+y^2 < 4 \end{cases}$

[Sol] Since $y > |x|$,

when $x \geqq 0$, $y > x$

when $x < 0$, $y > -x$



The boundaries are not includ

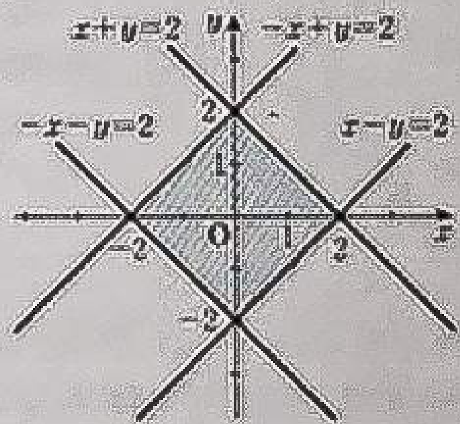
(3) $|x|+|y| \leqq 2$

[Sol] When $x \geqq 0$ and $y \geqq 0$, $x+y \leqq 2$

When $x \geqq 0$ and $y < 0$, $x-y \leqq 2$

When $x < 0$ and $y \geqq 0$, $-x+y \leqq 2$

When $x < 0$ and $y < 0$, $-x-y \leqq 2$



The boundaries are included.

M76b

(4) $\begin{cases} x^2+y^2-2x-8<0 \\ (x-y)(x+y+2)>0 \end{cases}$

[Sol] Since $x^2+y^2-2x-8<0$,

$$(x-1)^2+y^2<9$$

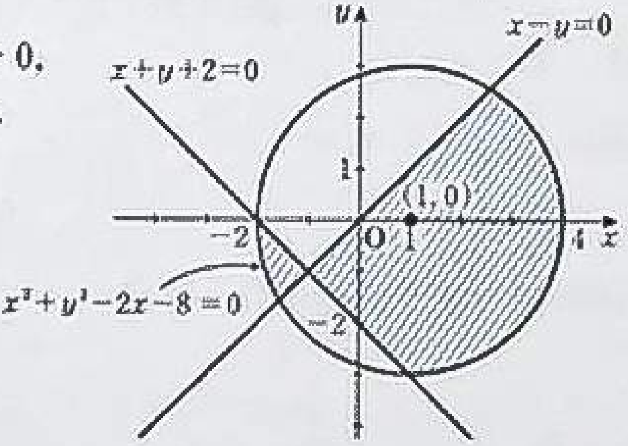
Also, since $(x-y)(x+y+2)>0$, when the given inequality is true, either

$$\begin{cases} x-y>0 \\ x+y+2>0 \end{cases}$$

or

$$\begin{cases} x-y<0 \\ x+y+2<0 \end{cases}$$

is true.

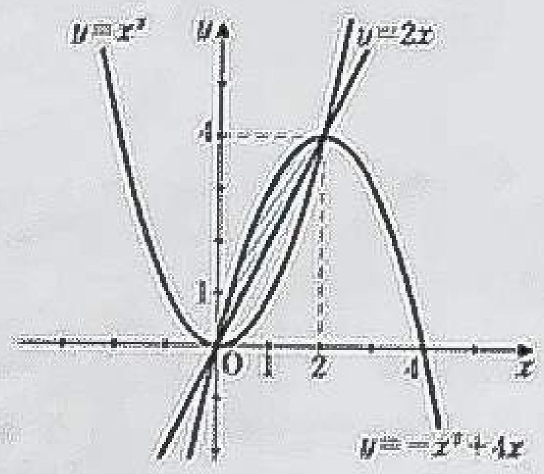


The boundaries are not included.

(5) $\begin{cases} y \geqq 2x \\ y \geqq x^2 \\ y \leqq -x^2+4x \end{cases}$

[Sol] Since $y \leqq -x^2+4x$,

$$y \leqq -(x-2)^2+4$$



The boundaries are included.

Regions

Name

Date / /

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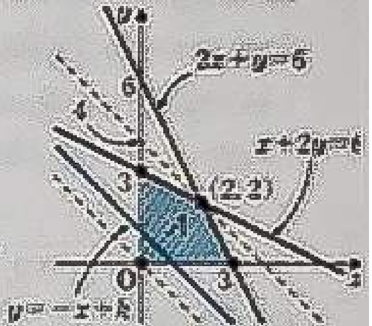
Time : to :

EX. Given that x and y satisfy four inequalities $x \geqq 0$, $y \geqq 0$, $2x+y \leqq 6$ and $x+2y \leqq 6$, find the maximum and minimum values of $x+y$ and the corresponding values of x and y.

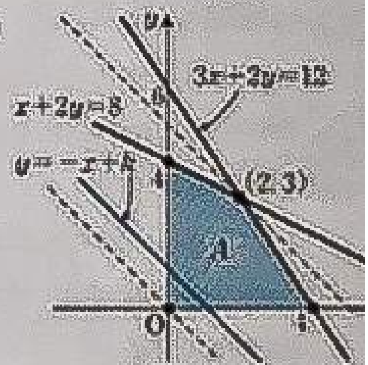


[Sol] Let A be the region represented by the four given inequalities. Region A lies inside and on the perimeter of the quadrilateral with four vertices $(0, 0)$, $(3, 0)$, $(2, 2)$ and $(0, 3)$.
Let $x+y=k$, i.e. $y=-x+k$ ···①.
① represents the line with slope -1 and y-intercept k.
In region A, when line ① passes through point $(2, 2)$, the value of k is maximized, and when line ① passes through point $(0, 0)$, the value of k is minimized.

Finding the maximum and minimum values of k in order for line ① and region A to have common points

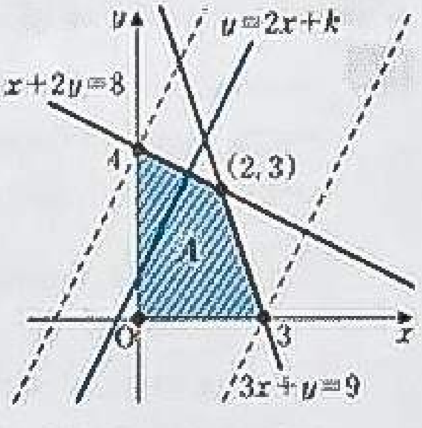
Therefore, the maximum value is 4, at $x=2$, $y=2$ and the minimum value is 0, at $x=0$, $y=0$.

1. Given that x and y satisfy four inequalities $x \geqq 0$, $y \geqq 0$, $x+2y \leqq 8$ and $3x+2y \leqq 12$, find the maximum and minimum values of $x+y$ and the corresponding values of x and y.



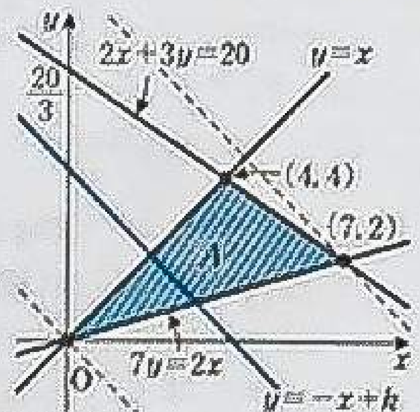
[Sol] Let A be the region represented by the four given inequalities. Region A lies inside and on the perimeter of the quadrilateral with four vertices $(0, 0)$, $(4, 0)$, $(2, 3)$ and $(0, 4)$.
Let $x+y=k$, i.e. $y=-x+k$ ···①.
① represents the line with slope -1 and y-intercept k.
In region A, when line ① passes through point $(2, 3)$, the value of k is maximized, and when line ① passes through point $(0, 0)$, the value of k minimized.
Therefore, the maximum value is 5, at $x=2$, $y=3$ and the minimum value is 0, at $x=0$, $y=0$.

2. Given that x and y satisfy four inequalities $x \geqslant 0$, $y \geqslant 0$, $x+2y \leqslant 8$ and $3x+y \leqslant 9$, find the maximum and minimum values of $-2x+y$ and the corresponding values of x and y.



[Sol] Let A be the region represented by the four given inequalities. Region A lies inside and on the perimeter of the quadrilateral with four vertices $(0, 0)$, $(3, 0)$, $(2, 3)$ and $(0, 4)$.
Let $-2x+y=k$, i.e. $y=2x+k$ ···①.
① represents the line with slope 2 and y-intercept k.
In region A, when line ① passes through point $(0, 4)$, the value of k is maximized, and when line ① passes through point $(3, 0)$, the value of k is minimized.
Therefore, **the maximum value is 4, at $x=0$, $y=4$ and the minimum value is -6, at $x=3$, $y=0$.**

3. Given that x and y satisfy three inequalities $y \leqslant x$, $7y \geqslant 2x$ and $2x+3y \leqslant 20$, find the maximum and minimum values of $x+y$ and the corresponding values of x and y.



[Sol] Let A be the region represented by the three given inequalities. Region A lies inside and on the perimeter of the triangle with three vertices $(0, 0)$, $(7, 2)$ and $(4, 4)$.
Let $x+y=k$, i.e. $y=-x+k$ ···①.
① represents the line with slope -1 and y-intercept k.
In region A, when line ① passes through point $(7, 2)$, the value of k is maximized, and when line ① passes through point $(0, 0)$, the value of k is minimized.
Therefore, **the maximum value is 9, at $x=7$, $y=2$ and the minimum value is 0, at $x=0$, $y=0$.**

Regions

Name

Date / /

Time : to :

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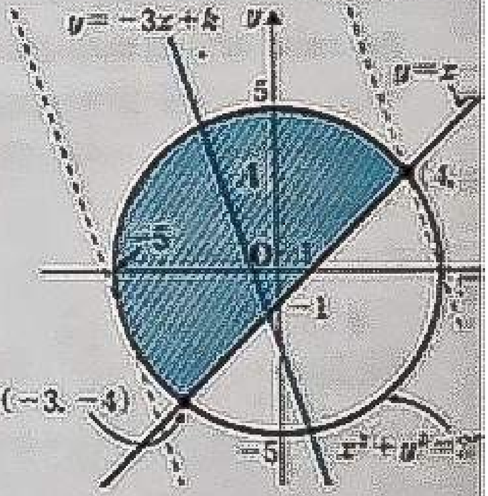
Mistakes: 0	—	—	—	—

1. Given that x and y satisfy two inequalities $y \geqq x-1$ and $x^2+y^2 \leqq 25$, find the maximum and minimum values of $3x+y$ and the corresponding values of x and y.

[Sol] Let A be the region represented by the two given inequalities.
Region A consists of the shaded region including the boundary.
Let $3x+y=k$, i.e. $y=-3x+k$ ···①.
① represents the line with slope -3 and y-intercept k.
In region A,
when line ① passes through point $(4, 3)$, the value of k is maximized.
Also, when line ① is tangent to circle $x^2+y^2=25$ ···②, the value of k is minimized.



From ① and ②,
$$x^2+(-3x+k)^2=25$$
So, for $10x^2-6kx+k^2-25=0$ ···③,
$$\frac{D}{4}=(-3k)^2-10(k^2-25)=0, \text{ i.e. } k=\pm 5\sqrt{10}$$
Therefore, the minimum value occurs when $k=-5\sqrt{10}$.
Substituting $k=-5\sqrt{10}$ into ③ and simplifying,

From the diagram, the minimum value occurs when $k=-5\sqrt{10}$, not $k=5\sqrt{10}$.

$$2x^2+6\sqrt{10}x+45=0$$
$$(\sqrt{2}x+3\sqrt{5})^2=0$$
$$\therefore\ x=-\frac{3\sqrt{10}}{2}$$
Substituting $x=-\dfrac{3\sqrt{10}}{2}$ and $k=-5\sqrt{10}$ into ①, $y=-\dfrac{\sqrt{10}}{2}$

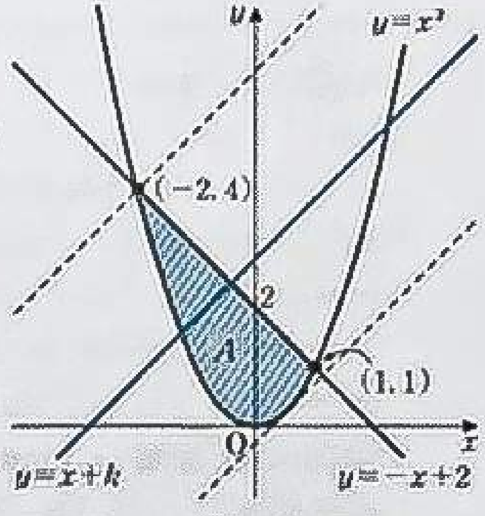
Thus, the maximum value is 15, at $x=4$, $y=3$ and

the minimum value is $-5\sqrt{10}$, at $x=-\dfrac{3\sqrt{10}}{2}$, $y=-\dfrac{\sqrt{10}}{2}$.

M78b

2. Given that x and y satisfy two inequalities $y \geqslant x^2$ and $y \leqslant -x+2$, find the maximum and minimum values of $y-x$ and the corresponding values of x and y.

[Sol] Let A be the region represented by the two given inequalities.
Region A consists of the shaded region including the boundary.
Let $y-x=k$, i.e. $y=x+k$ ⋯①. ① represents the line with slope 1 and y-intercept k.
In region A,
when line ① passes through point $(-2, 4)$, the value of k is maximized.
Also, when line ① is tangent to parabola $y=x^2$ ⋯②, the value of k is minimized.



From ① and ②,

$$x+k=x^2$$

So, for $x^2-x-k=0$ ⋯③,

$$D=(-1)^2-4\cdot1\cdot(-k)=0, \text{ i.e. } k=-\frac{1}{4}$$

Therefore, the minimum value occurs when $k=-\frac{1}{4}$.

Substituting $k=-\frac{1}{4}$ into ③,

$$x^2-x+\frac{1}{4}=0$$

$$\left(x-\frac{1}{2}\right)^2=0$$

$$\therefore\ x=\frac{1}{2}$$

Substituting $x=\frac{1}{2}$ and $k=-\frac{1}{4}$ into ①, $y=\frac{1}{4}$

Thus, the maximum value is 6, at $x=-2$, $y=4$ and
the minimum value is $-\frac{1}{4}$, at $x=\frac{1}{2}$, $y=\frac{1}{4}$.

Regions

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~

1. A bakery sells cakes and cookies. The amount of flour and eggs required to bake a cake or a bag of cookies and the amount of stock are listed in the table. A cake sells for 10 dollars and a bag of cookies sells for 20 dollars. Given 9000g of flour and 80 eggs, how many cakes and bags of cookies ar needed to be baked in order to maximize sales?

	flour	eggs
cake	100 g	2
cookies	300 g	1
stock	9000 g	80

[Sol] Let the number of cakes and the number of bags of cookies be x and y respectively.

$x \geqq 0$ $\cdots$①, $y \geqq 0$ $\cdots$②

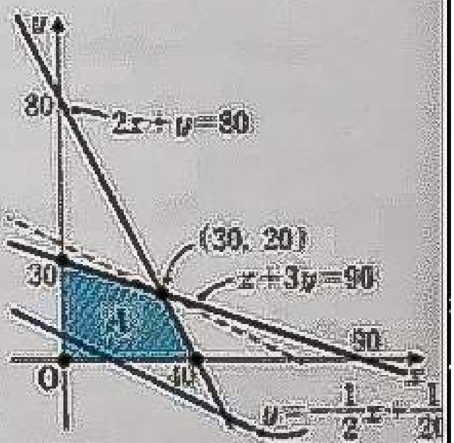
For the stock of flour,

since $100x+300y \leqq 9000$,

$x+3y \leqq 90$ $\cdots$③

For the stock of eggs,

$2x+y \leqq 80$ $\cdots$④



Let A be the region represented by ①~④.

Region A lies inside and on the perimeter of the quadrilateral with fo vertices (0, 0), (40, 0), (30, 20) and (0, 30).

Since the sales are $(10x+20y)$ dollars,

let $10x+20y=k$, i.e. $y=-\frac{1}{2}x+\frac{1}{20}k$ $\cdots$⑤.

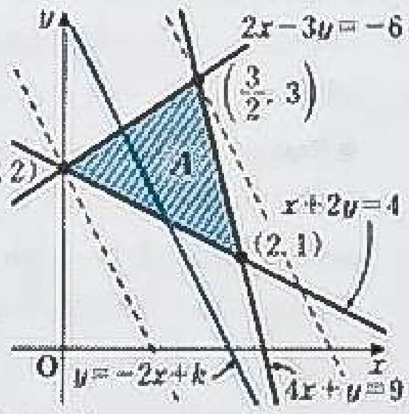
Then, ⑤ represents the line with slope $-\frac{1}{2}$ and y-intercept $\frac{1}{20}k$.

In region A, when line ⑤ passes through point (30, 20), the value of k maximized.

Therefore, 30 cakes and 20 bags of cookies are needed to be baked.

M79b

2. Given moving point $P(x, y)$ in the region represented by $4x+y \leqq 9$, $x+2y \geqq 4$ and $2x-3y \geqq -6$, find the maximum and minimum values of $2x+y$ and x^2+y^2, and the corresponding values of x and y.



[Sol] Let A be the region represented by the three given inequalities. Region A lies inside and on the perimeter of the triangle with three vertices $(2, 1)$, $\left(\frac{3}{2}, 3\right)$ and $(0, 2)$.

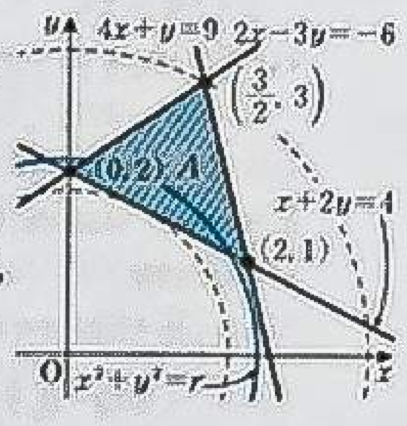
Let $2x+y=k$, i.e. $y=-2x+k$ ⋯①.

① represents the line with slope -2 and y-intercept k.

In region A, when line ① passes through point $\left(\frac{3}{2}, 3\right)$, the value of k is maximized, and when line ① passes through point $(0, 2)$, the value of k is minimized.

Therefore, the maximum value is 6, at $x=\frac{3}{2}$, $y=3$ and

the minimum value is 2, at $x=0$, $y=2$.



Also, let $x^2+y^2=r$ ⋯②. ② represents the circle with its center at the origin and radius $\sqrt{r}$.

In region A, when circle ② passes through point $\left(\frac{3}{2}, 3\right)$, the value of r is maximized.

When circle ② is tangent to line $x+2y=4$ ⋯③, the value of r is minimized.

From ② and ③,

$$x^2+\left(-\frac{1}{2}x+2\right)^2=r$$

So, for $\frac{5}{4}x^2-2x+4-r=0$ ⋯④,

$$\frac{D}{4}=(-1)^2-\frac{5}{4}\cdot(4-r)=0, \text{ i.e. } r=\frac{16}{5} \quad \cdots⑤$$

From ④ and ⑤,

$$25x^2-40x+16=0$$

$$(5x-4)^2=0$$

$$\therefore x=\frac{4}{5} \quad \cdots⑥$$

From ③ and ⑥, $y=\frac{8}{5}$

Thus, the maximum value is $\frac{45}{4}$, at $x=\frac{3}{2}$, $y=3$ and

the minimum value is $\frac{16}{5}$, at $x=\frac{4}{5}$, $y=\frac{8}{5}$.

Regions

Name

Date / /

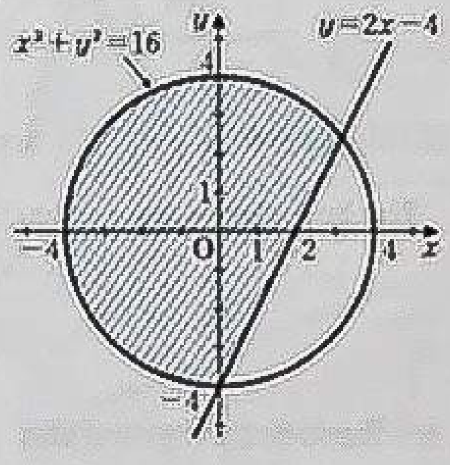
100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	2	3~

Time : to :

1. Shade the region satisfying each given system of inequalities. ➡ M73.1

(1) $\begin{cases} x^2+y^2 \leqq 16 \\ y \geqq 2x-4 \end{cases}$

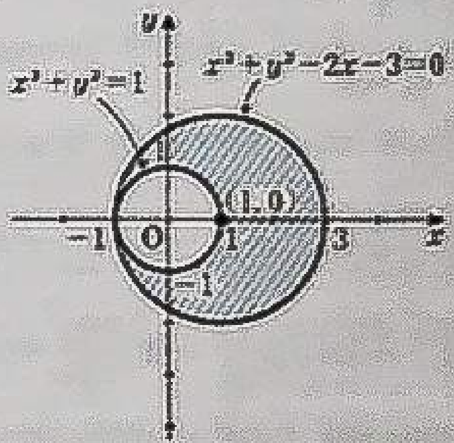
[Sol]



The boundaries are included.

(3) $\begin{cases} x^2+y^2 \geqq 1 \\ x^2+y^2-2x-3 \leqq 0 \end{cases}$

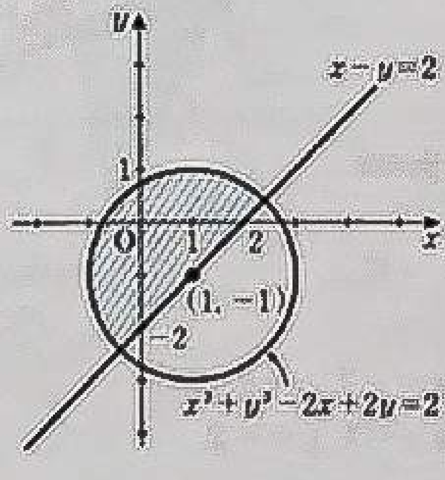
[Sol]



The boundaries are included.

(2) $\begin{cases} x^2+y^2-2x+2y < 2 \\ x-y < 2 \end{cases}$

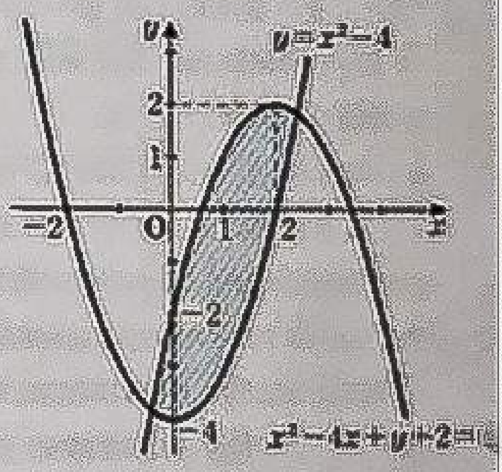
[Sol]



The boundaries are not included.

(4) $\begin{cases} y > x^2-4 \\ x^2-4x+y+2 < 0 \end{cases}$

[Sol]



The boundaries are not included.

2. Shade the region satisfying the given inequality. ➡ M75

$$(y+x^2)(y-x^2+2)>0$$

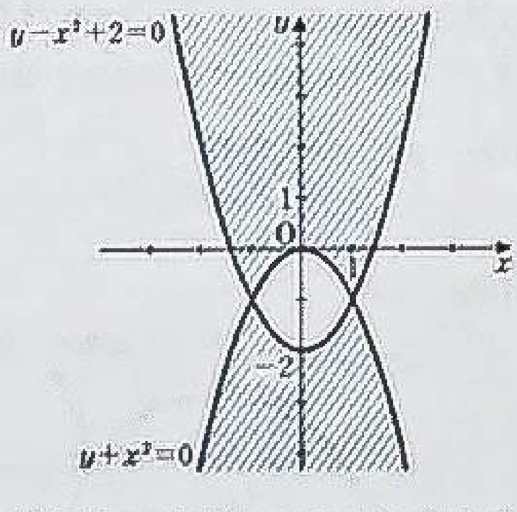
[Sol] When the given inequality is true, either

$$\begin{cases} y+x^2>0 \\ y-x^2+2>0 \end{cases}$$

or

$$\begin{cases} y+x^2<0 \\ y-x^2+2<0 \end{cases}$$

is true.



The boundaries are not included.

3. Given that x and y satisfy three inequalities $x+2y \leqq 6$, $2x+y \geqq 0$ and $4x-y \leqq 6$, find the maximum and minimum values of $-x+y$ and the corresponding values of x and y. ➡ M77

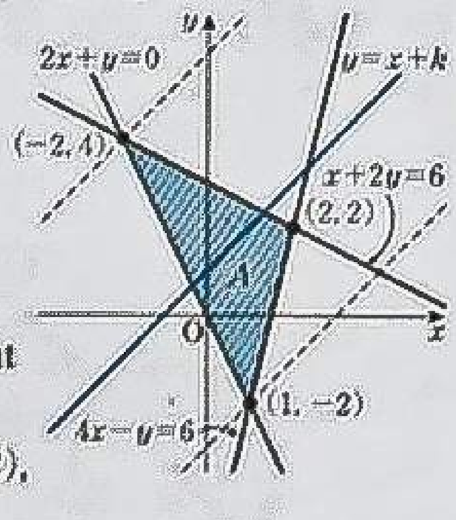
[Sol] Let A be the region represented by the three given inequalities.

Region A lies inside and on the perimeter of the triangle with three vertices $(-2, 4)$, $(1, -2)$ and $(2, 2)$.

Let $-x+y=k$, i.e. $y=x+k$ ···①.

① represents the line with slope 1 and y-intercept k.

In region A, when line ① passes through point $(-2, 4)$, the value of k is maximized, and when line ① passes through point $(1, -2)$, the value of k is minimized.



Therefore, the maximum value is 6, at $x=-2$, $y=4$ and the minimum value is -3, at $x=1$, $y=-2$.

Trigonometric Ratios 1

Name ____________

Date / /

Time : to :

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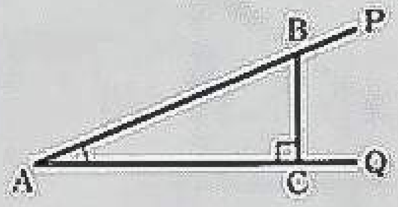
Given $\angle PAQ$, drop a perpendicular BC from point B on AP to AQ and form right-angled triangle ABC. When A represents the size of $\angle A$,



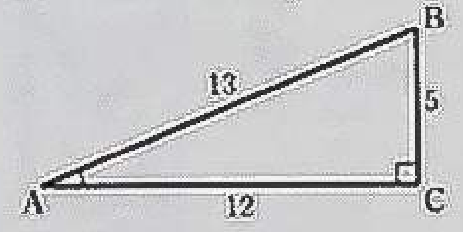
$\frac{BC}{AB}$ is called the ***sine of A*** and expressed as $\sin A$.

$\frac{AC}{AB}$ is called the ***cosine of A*** and expressed as $\cos A$

$\frac{BC}{AC}$ is called the ***tangent of A*** and expressed as $\tan A$

Given a right-angled triangle ABC, find the values of $\sin A$, $\cos A$ and $\tan A$.

Ex.

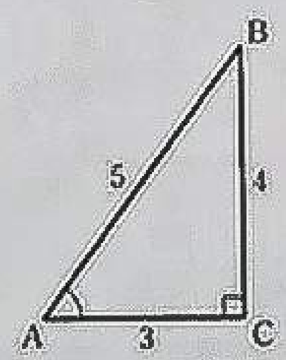


[Sol] $\sin A=\frac{5}{13}$ ← From $\sin A=\frac{BC}{AB}$

$\cos A=\frac{12}{13}$ ← From $\cos A=\frac{AC}{AB}$

$\tan A=\frac{5}{12}$ ← From $\tan A=\frac{BC}{AC}$

(1)

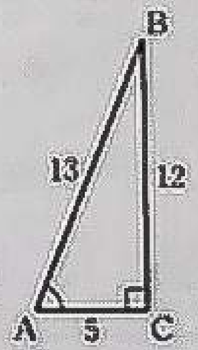


[Sol] $\sin A=\frac{4}{5}$

$\cos A=\frac{3}{5}$

$\tan A=\frac{4}{3}$

(2)



[Sol] $\sin A=\frac{12}{13}$

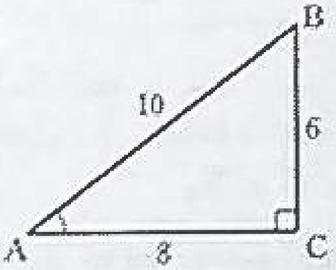
$\cos A=\frac{5}{13}$

$\tan A=\frac{12}{5}$

The sine, cosine and tangent are called ***trigonometric ratios***.
$\sin A$, $\cos A$ and $\tan A$ are the ratio values defined by the size of A.

M81b

(3)

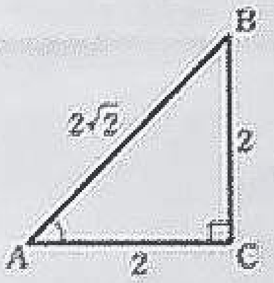


[Sol] $\sin A = \dfrac{\boxed{6}}{10} = \dfrac{3}{5}$

$\cos A = \dfrac{8}{10} = \dfrac{4}{5}$

$\tan A = \dfrac{6}{8} = \dfrac{3}{4}$

(4)

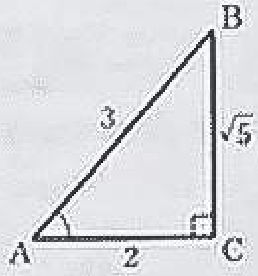


[Sol] $\sin A = \dfrac{2}{2\sqrt{2}} = \dfrac{\sqrt{2}}{2} \quad \left[= \dfrac{1}{\sqrt{2}}\right]$

$\cos A = \dfrac{2}{2\sqrt{2}} = \dfrac{\sqrt{2}}{2} \quad \left[= \dfrac{1}{\sqrt{2}}\right]$

$\tan A = \dfrac{2}{2} = 1$

(5)



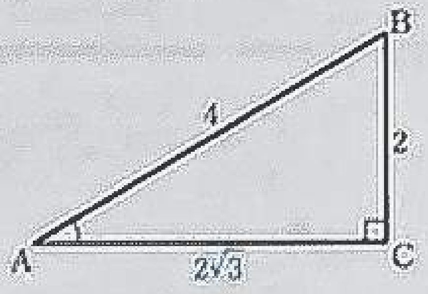
[Sol] $AB = \sqrt{\boxed{2}^2 + (\boxed{\sqrt{5}})^2} = 3$

$\sin A = \dfrac{\sqrt{5}}{3}$

$\cos A = \dfrac{2}{3}$

$\tan A = \dfrac{\sqrt{5}}{2}$

(6)



[Sol] $AC = \sqrt{4^2 - 2^2} = 2\sqrt{3}$

$\sin A = \dfrac{2}{4} = \dfrac{1}{2}$

$\cos A = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2}$

$\tan A = \dfrac{2}{2\sqrt{3}} = \dfrac{\sqrt{3}}{3} \quad \left[= \dfrac{1}{\sqrt{3}}\right]$

It is possible to use fractions, such as $\dfrac{1}{\sqrt{2}}$, $\dfrac{1}{\sqrt{3}}$ and $\dfrac{2}{\sqrt{5}}$, without rationalizing the denominators.

Trigonometric Ratios 1

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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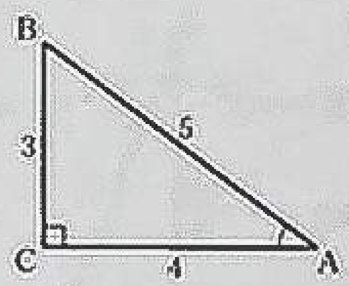
Given a right-angled triangle ABC, find the values of $\sin A$, $\cos A$ and $\tan A$.

Ex.

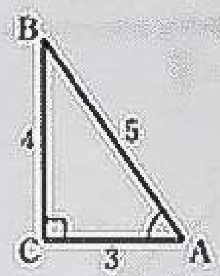


[Sol] $\sin A=\frac{3}{5}$

$\cos A=\frac{4}{5}$

$\tan A=\frac{3}{4}$

(1)

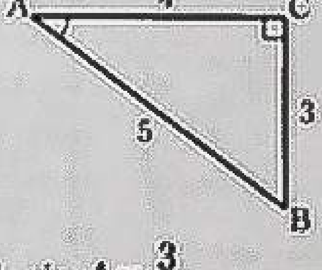


[Sol] $\sin A=\frac{4}{5}$

$\cos A=\frac{3}{5}$

$\tan A=\frac{4}{3}$

(2)

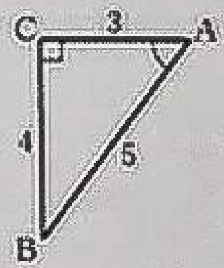


[Sol] $\sin A=\frac{3}{5}$

$\cos A=\frac{4}{5}$

$\tan A=\frac{3}{4}$

(3)

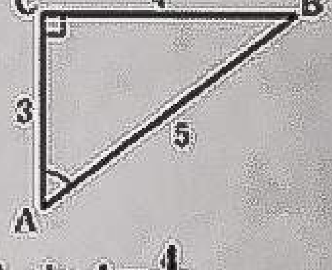


[Sol] $\sin A=\frac{4}{5}$

$\cos A=\frac{3}{5}$

$\tan A=\frac{4}{3}$

(4)



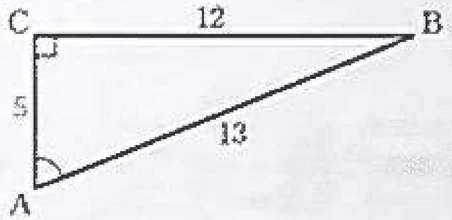
[Sol] $\sin A=\frac{4}{5}$

$\cos A=\frac{3}{5}$

$\tan A=\frac{4}{3}$

M82b

(5)



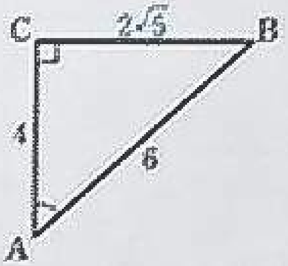
[Sol] $AC=\sqrt{13^2-12^2}=5$

$$\sin A=\frac{12}{13}$$

$$\cos A=\frac{5}{13}$$

$$\tan A=\frac{12}{5}$$

(6)



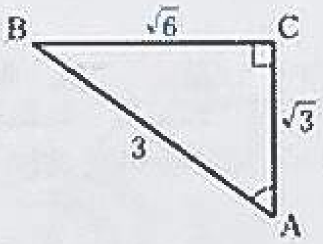
[Sol] $BC=\sqrt{6^2-4^2}=2\sqrt{5}$

$$\sin A=\frac{2\sqrt{5}}{6}=\frac{\sqrt{5}}{3}$$

$$\cos A=\frac{4}{6}=\frac{2}{3}$$

$$\tan A=\frac{2\sqrt{5}}{4}=\frac{\sqrt{5}}{2}$$

(7)



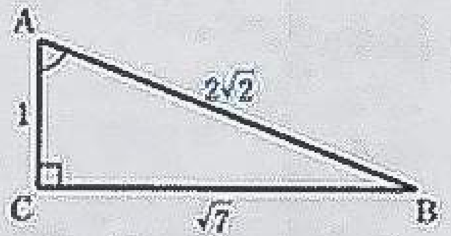
[Sol] $BC=\sqrt{3^2-(\sqrt{3})^2}=\sqrt{6}$

$$\sin A=\frac{\sqrt{6}}{3}$$

$$\cos A=\frac{\sqrt{3}}{3}$$

$$\tan A=\frac{\sqrt{6}}{\sqrt{3}}=\sqrt{2}$$

(8)



[Sol] $AB=\sqrt{1^2+(\sqrt{7})^2}=2\sqrt{2}$

$$\sin A=\frac{\sqrt{7}}{2\sqrt{2}}=\frac{\sqrt{14}}{4}$$

$$\cos A=\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}\quad\left[=\frac{1}{2\sqrt{2}}\right]$$

$$\tan A=\frac{\sqrt{7}}{1}=\sqrt{7}$$

Trigonometric Ratios 1

Name

Date / /

Time : to :

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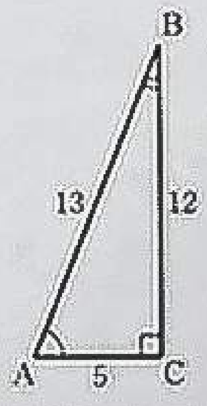
[mistakes] 0	—	—	1	2~

Given a right-angled triangle ABC, find the values of sin, cos and tan.

Ex.



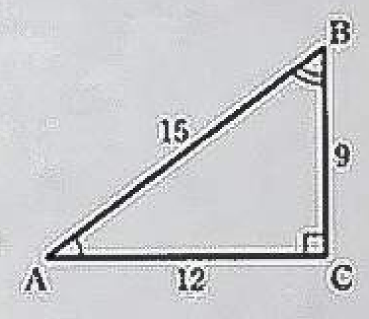
[Sol]

$\sin A=\frac{12}{13}$ $\sin B=\frac{5}{13}$

$\cos A=\frac{5}{13}$ $\cos B=\frac{12}{13}$

$\tan A=\frac{12}{5}$ $\tan B=\frac{5}{12}$

(1)



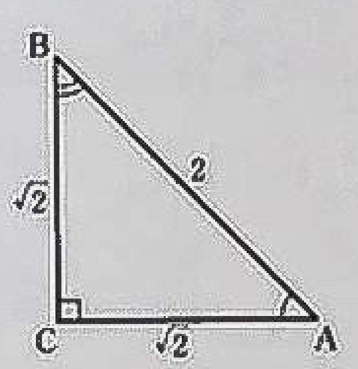
[Sol]

$\sin A=\frac{9}{15}=\frac{3}{5}$ $\sin B=\frac{12}{15}=\frac{4}{5}$

$\cos A=\frac{12}{15}=\frac{4}{5}$ $\cos B=\frac{9}{15}=\frac{3}{5}$

$\tan A=\frac{9}{12}=\frac{3}{4}$ $\tan B=\frac{12}{9}=\frac{4}{3}$

(2)



[Sol]

$\sin A=\frac{\sqrt{2}}{2}$ $\sin B=\frac{\sqrt{2}}{2}$

$\cos A=\frac{\sqrt{2}}{2}$ $\cos B=\frac{\sqrt{2}}{2}$

$\tan A=\frac{\sqrt{2}}{\sqrt{2}}=1$ $\tan B=\frac{\sqrt{2}}{\sqrt{2}}=1$

M83b

(3)

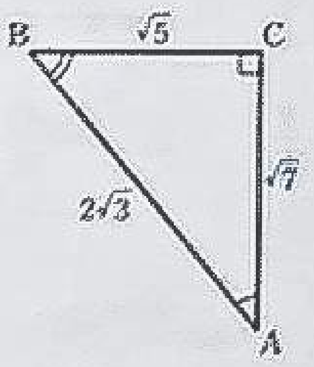
[Sol] $AB=\sqrt{2^2+6^2}=2\sqrt{10}$

$\sin A=\dfrac{6}{2\sqrt{10}}=\dfrac{3\sqrt{10}}{10}$ $\left[=\dfrac{3}{\sqrt{10}}\right]$ $\sin B=\dfrac{2}{2\sqrt{10}}=\dfrac{\sqrt{10}}{10}$ $\left[=\dfrac{1}{\sqrt{10}}\right]$

$\cos A=\dfrac{2}{2\sqrt{10}}=\dfrac{\sqrt{10}}{10}$ $\left[=\dfrac{1}{\sqrt{10}}\right]$ $\cos B=\dfrac{6}{2\sqrt{10}}=\dfrac{3\sqrt{10}}{10}$ $\left[=\dfrac{3}{\sqrt{10}}\right]$

$\tan A=\dfrac{6}{2}=3$ $\tan B=\dfrac{2}{6}=\dfrac{1}{3}$

(4)



[Sol] $AC=\sqrt{(2\sqrt{3})^2-(\sqrt{5})^2}=\sqrt{7}$

$\sin A=\dfrac{\sqrt{5}}{2\sqrt{3}}=\dfrac{\sqrt{15}}{6}$ $\sin B=\dfrac{\sqrt{7}}{2\sqrt{3}}=\dfrac{\sqrt{21}}{6}$

$\cos A=\dfrac{\sqrt{7}}{2\sqrt{3}}=\dfrac{\sqrt{21}}{6}$ $\cos B=\dfrac{\sqrt{5}}{2\sqrt{3}}=\dfrac{\sqrt{15}}{6}$

$\tan A=\dfrac{\sqrt{5}}{\sqrt{7}}=\dfrac{\sqrt{35}}{7}$ $\tan B=\dfrac{\sqrt{7}}{\sqrt{5}}=\dfrac{\sqrt{35}}{5}$

Trigonometric Ratios 1

Name

Date / /

Time : to :

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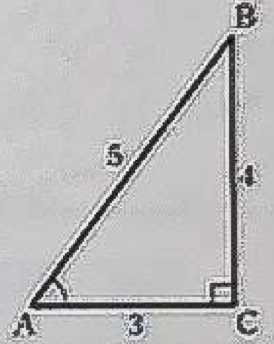
Ex. Given $\sin A = \frac{4}{5}$, find the values of $\cos A$ and $\tan A$. $(0° < A < 90°)$

[Sol] $AC = \sqrt{5^2 - 4^2} = 3$

$\cos A = \frac{3}{5}$

$\tan A = \frac{4}{3}$

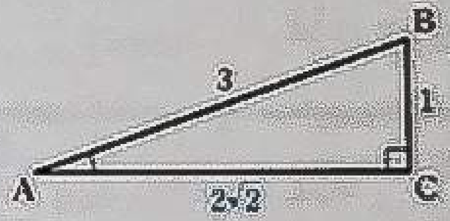


1. Given $\sin A = \frac{1}{3}$, find the values of $\cos A$ and $\tan A$. $(0° < A < 90°)$

[Sol] $AC = \sqrt{3^2 - 1^2} = 2\sqrt{2}$

$\cos A = \frac{2\sqrt{2}}{3}$

$\tan A = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \quad \left[= \frac{1}{2\sqrt{2}} \right]$

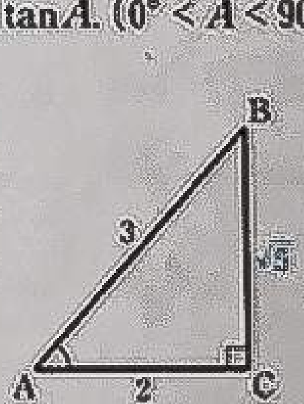


2. Given $\cos A = \frac{2}{3}$, find the values of $\sin A$ and $\tan A$. $(0° < A < 90°)$

[Sol] $BC = \sqrt{3^2 - 2^2} = \sqrt{5}$

$\sin A = \frac{\sqrt{5}}{3}$

$\tan A = \frac{\sqrt{5}}{2}$



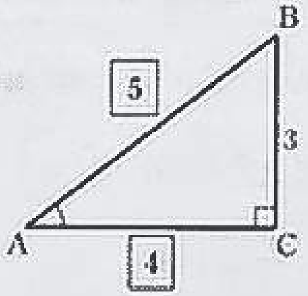
M84b

3. Given $\cos A = \frac{4}{5}$, fill in the blanks in the diagram below and find the values of $\sin A$ and $\tan A$. $(0° < A < 90°)$

[Sol] $BC = \sqrt{5^2 - 4^2} = 3$

$\sin A = \frac{3}{5}$

$\tan A = \frac{3}{4}$

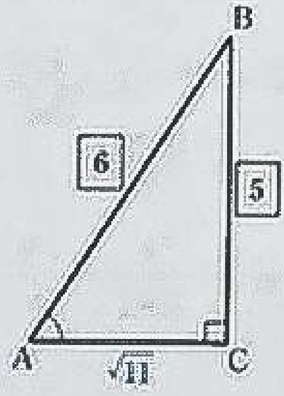


4. Given $\sin A = \frac{5}{6}$, fill in the blanks in the diagram below and find the values of $\cos A$ and $\tan A$. $(0° < A < 90°)$

[Sol] $AC = \sqrt{6^2 - 5^2} = \sqrt{11}$

$\cos A = \frac{\sqrt{11}}{6}$

$\tan A = \frac{5}{\sqrt{11}} = \frac{5\sqrt{11}}{11}$ $\left[= \frac{5}{\sqrt{11}} \right]$

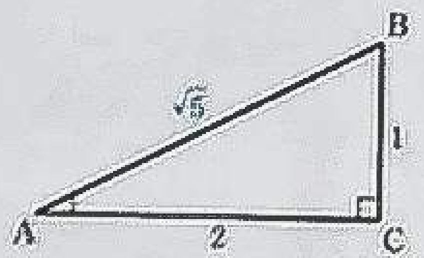


5. Given $\tan A = \frac{1}{2}$, find the values of $\sin A$ and $\cos A$ by drawing the diagram. $(0° < A < 90°)$

[Sol] $AB = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\sin A = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$ $\left[= \frac{1}{\sqrt{5}} \right]$

$\cos A = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$ $\left[= \frac{2}{\sqrt{5}} \right]$



Trigonometric Ratios 1

Name

Date / /

Time : to :

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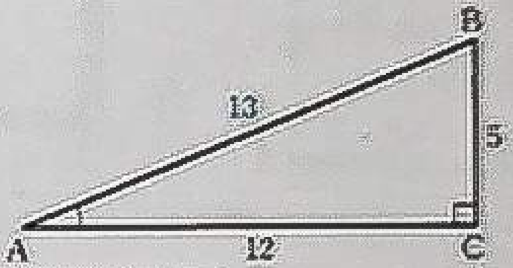
(mistakes) 0	—	—	—	—

Ex. Given $\tan A=\frac{5}{12}$, find the values of $\frac{\sin A}{\cos A}$ and $\sin^2 A+\cos^2 A$.

[Sol] $AB=\sqrt{12^2+5^2}=13$



$\sin A=\frac{5}{13}$

$\cos A=\frac{12}{13}$

$\therefore \frac{\sin A}{\cos A}=\frac{5}{13}\div\frac{12}{13}$ ← $\frac{\frac{5}{13}}{\frac{12}{13}}=\frac{5}{13}\div\frac{12}{13}$

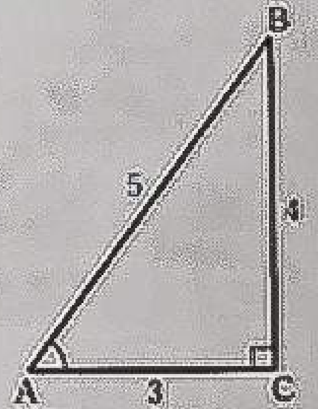
$=\frac{5}{12}$

$\sin^2 A+\cos^2 A=\left(\frac{5}{13}\right)^2+\left(\frac{12}{13}\right)^2$

$=1$

1. Given $\tan A=\frac{4}{3}$, find the values of $\frac{\sin A}{\cos A}$ and $\sin^2 A+\cos^2 A$.

[Sol] $AB=\sqrt{3^2+4^2}=5$



$\sin A=\frac{4}{5}$

$\cos A=\frac{3}{5}$

$\therefore \frac{\sin A}{\cos A}=\frac{4}{5}\div\frac{3}{5}$

$=\frac{4}{3}$

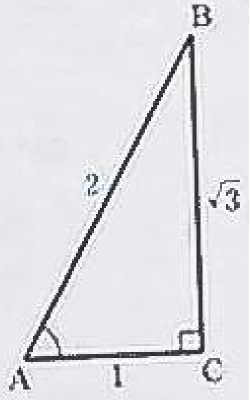
$\sin^2 A+\cos^2 A=\left(\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2$

$=1$

$(\sin A)^2$, $(\cos A)^2$ and $(\tan A)^2$ are expressed as $\sin^2 A$, $\cos^2 A$ and $\tan^2 A$ respectively.

M85b

2. Given $\tan A=\sqrt{3}$, find the values of $\dfrac{\sin A}{\cos A}$ and $\sin^2 A+\cos^2 A$.



[Sol] $AB=\sqrt{1^2+(\sqrt{3})^2}=2$

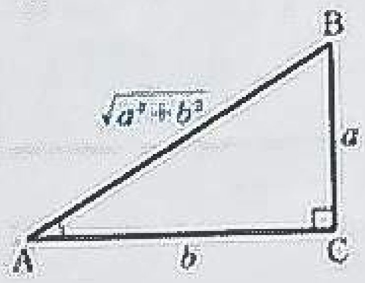
$$\sin A=\frac{\sqrt{3}}{2}$$

$$\cos A=\frac{1}{2}$$

$$\therefore \frac{\sin A}{\cos A}=\frac{\sqrt{3}}{2}\div\frac{1}{2}=\sqrt{3}$$

$$\sin^2 A+\cos^2 A=\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2=1$$

3. Given $\tan A=\dfrac{a}{b}$ $(a>0,\ b>0)$, find the values of $\dfrac{\sin A}{\cos A}$ and $\sin^2 A+\cos^2 A$.



[Sol] $AB=\sqrt{\boxed{a^2+b^2}}$

$$\sin A=\frac{\boxed{a}}{\sqrt{\boxed{a^2+b^2}}}$$

$$\cos A=\frac{\boxed{b}}{\sqrt{\boxed{a^2+b^2}}}$$

$$\therefore \frac{\sin A}{\cos A}=\frac{\boxed{a}}{\sqrt{\boxed{a^2+b^2}}}\div\frac{\boxed{b}}{\sqrt{\boxed{a^2+b^2}}}=\frac{\boxed{a}}{\boxed{b}}.$$

$$\sin^2 A+\cos^2 A=\left(\frac{\boxed{a}}{\sqrt{\boxed{a^2+b^2}}}\right)^2+\left(\frac{\boxed{b}}{\sqrt{\boxed{a^2+b^2}}}\right)^2=\boxed{1}$$

From the solutions to question 3, the identities $\tan A=\dfrac{\sin A}{\cos A}$ and $\sin^2 A+\cos^2 A=1$ are true.

Trigonometric Ratios 1

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	1	2~

Trigonometric Identities 1

$$\tan A = \frac{\sin A}{\cos A}$$

$$\sin^2 A + \cos^2 A = 1$$

Ex. Given $\sin A = \frac{2}{3}$, find the values of $\cos A$ and $\tan A$. $(0° < A < 90°)$

[Sol] $\cos^2 A = 1 - \left(\frac{2}{3}\right)^2 = \frac{5}{9}$ ← Since $\sin^2 A + \cos^2 A = 1$, $\cos^2 A = 1 - \sin^2 A$

Since $\cos A > 0$, $\cos A = \frac{\sqrt{5}}{3}$

Also, $\tan A = \frac{2}{3} \div \frac{\sqrt{5}}{3} = \frac{2\sqrt{5}}{5}$ ← $\tan A = \frac{\sin A}{\cos A}$

1. Given $\sin A = \frac{2}{5}$, find the values of $\cos A$ and $\tan A$. $(0° < A < 90°)$

[Sol] $\cos^2 A = 1 - \left(\frac{2}{5}\right)^2 = \frac{21}{25}$

Since $\cos A > 0$, $\cos A = \frac{\sqrt{21}}{5}$

Also, $\tan A = \frac{2}{5} \div \frac{\sqrt{21}}{5} = \frac{2\sqrt{21}}{21}$ $\left[= \frac{2}{\sqrt{21}} \right]$

M86b

2. Given $\cos A=\frac{1}{4}$, find the values of $\sin A$ and $\tan A$. $(0^\circ < A < 90^\circ)$

[Sol] $\sin^2 A=1-\left(\frac{1}{4}\right)^2=\frac{15}{16}$

Since $\sin A>0$, $\sin A=\frac{\sqrt{15}}{4}$

Also, $\tan A=\frac{\sqrt{15}}{4}\div\frac{1}{4}=\sqrt{15}$

3. Given $\sin A=\frac{5}{13}$, find the values of $\cos A$ and $\tan A$. $(0^\circ < A < 90^\circ)$

[Sol] $\cos^2 A=1-\left(\frac{5}{13}\right)^2=\frac{144}{169}$

Since $\cos A>0$, $\cos A=\frac{12}{13}$

Also, $\tan A=\frac{5}{13}\div\frac{12}{13}=\frac{5}{12}$

4. Given $\cos A=\frac{5}{7}$, find the values of $\sin A$ and $\tan A$. $(0^\circ < A < 90^\circ)$

[Sol] $\sin^2 A=1-\left(\frac{5}{7}\right)^2=\frac{24}{49}$

Since $\sin A>0$, $\sin A=\frac{2\sqrt{6}}{7}$

Also, $\tan A=\frac{2\sqrt{6}}{7}\div\frac{5}{7}=\frac{2\sqrt{6}}{5}$

Trigonometric Ratios 1

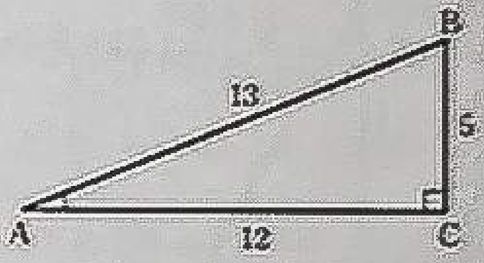
Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(Mistakes) 0	—	1	—	2

1. Given three sides of the right-angled triangle as shown below, find the valu
of the following expressions.

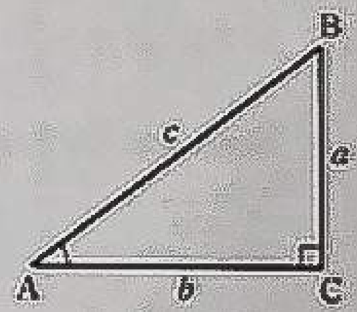


(1) $1+\tan^2 A=1+\left(\boxed{\dfrac{5}{12}}\right)^2$

$=\dfrac{169}{144}$

(2) $\dfrac{1}{\cos^2 A}=\dfrac{1}{\left(\boxed{\dfrac{12}{13}}\right)^2}=\dfrac{169}{144}$

2. Given three sides of the right-angled triangle as shown below, find the valu
of the following expressions.



(1) $1+\tan^2 A=1+\left(\boxed{\dfrac{a}{b}}\right)^2$

$=\dfrac{\boxed{a^2+b^2}}{\boxed{b^2}}$

The Pythagorean Theorem

$=\dfrac{\boxed{c^2}}{\boxed{b^2}}$

(2) $\dfrac{1}{\cos^2 A}=\dfrac{1}{\left(\boxed{\dfrac{b}{c}}\right)^2}$

$=\dfrac{\boxed{c^2}}{\boxed{b^2}}$

From the solutions to question 2, the identity $1+\tan^2 A=\dfrac{1}{\cos^2 A}$ is true.

M87b

Trigonometric Identity II

$$1+\tan^2 A=\frac{1}{\cos^2 A}$$

EX Given $\tan A=2\sqrt{2}$, find the values of $\cos A$ and $\sin A$. $(0° < A < 90°)$

[Sol] $\frac{1}{\cos^2 A}=1+(2\sqrt{2})^2=9$ ⟵ $1+\tan^2 A=\frac{1}{\cos^2 A}$

$\therefore \cos^2 A=\frac{1}{9}$

Since $\cos A>0$, $\cos A=\frac{1}{3}$

Also, $\sin A=2\sqrt{2}\cdot\frac{1}{3}=\frac{2\sqrt{2}}{3}$ ⟵ Since $\tan A=\frac{\sin A}{\cos A}$, $\sin A=\tan A\cos A$

3. Given $\tan A=\frac{1}{2}$, find the values of $\cos A$ and $\sin A$. $(0° < A < 90°)$

[Sol] $\frac{1}{\cos^2 A}=1+\left(\frac{1}{2}\right)^2=\frac{5}{4}$

$\therefore \cos^2 A=\frac{4}{5}$

Since $\cos A>0$, $\cos A=\frac{2\sqrt{5}}{5}$ $\left[=\frac{2}{\sqrt{5}}\right]$

Also, $\sin A=\frac{1}{2}\cdot\frac{2\sqrt{5}}{5}=\frac{\sqrt{5}}{5}$ $\left[=\frac{1}{\sqrt{5}}\right]$

4. Given $\tan A=3$, find the values of $\cos A$ and $\sin A$. $(0° < A < 90°)$

[Sol] $\frac{1}{\cos^2 A}=1+3^2=10$

$\therefore \cos^2 A=\frac{1}{10}$

Since $\cos A>0$, $\cos A=\frac{\sqrt{10}}{10}$ $\left[=\frac{1}{\sqrt{10}}\right]$

Also, $\sin A=3\cdot\frac{\sqrt{10}}{10}=\frac{3\sqrt{10}}{10}$ $\left[=\frac{3}{\sqrt{10}}\right]$

Trigonometric Ratios 1

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	2	3

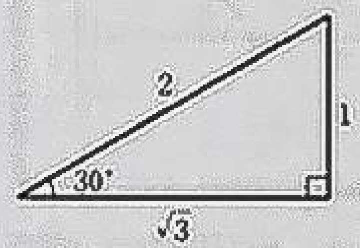
Name

Date / /

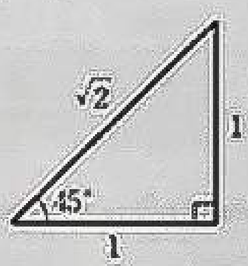
Time : to :

1. Using the right-angled triangles below, find the trigonometric ratios of 30°, 45° and 60°.

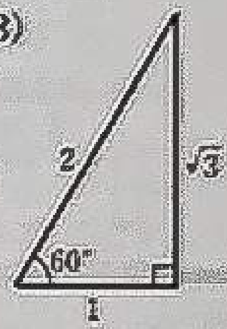
(1)



(2)



(3)



$$\begin{cases} \sin 30^\circ = \dfrac{1}{2} \\ \cos 30^\circ = \dfrac{\sqrt{3}}{2} \\ \tan 30^\circ = \dfrac{\sqrt{3}}{3} \quad \left[= \dfrac{1}{\sqrt{3}} \right] \end{cases}$$

$$\begin{cases} \sin 45^\circ = \dfrac{\sqrt{2}}{2} \quad \left[= \dfrac{1}{\sqrt{2}} \right] \\ \cos 45^\circ = \dfrac{\sqrt{2}}{2} \quad \left[= \dfrac{1}{\sqrt{2}} \right] \\ \tan 45^\circ = 1 \end{cases}$$

$$\begin{cases} \sin 60^\circ = \dfrac{\sqrt{3}}{2} \\ \cos 60^\circ = \dfrac{1}{2} \\ \tan 60^\circ = \sqrt{3} \end{cases}$$

2. Find the values of the following expressions.

Ex.

$$\sin 30^\circ \cos 60^\circ = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

(1) $\sin 45^\circ \cos 45^\circ = \dfrac{\sqrt{2}}{2} \cdot \dfrac{\sqrt{2}}{2} = \dfrac{1}{2}$

(2) $\cos 30^\circ \tan 60^\circ = \dfrac{\sqrt{3}}{2} \cdot \sqrt{3} = \dfrac{3}{2}$

(3) $\sin 60^\circ - \tan 30^\circ = \dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{3}}{3} = \dfrac{\sqrt{3}}{6}$

(4) $\tan 45^\circ - \cos 60^\circ = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

M88b

Ex Given right-angled triangle ABC with $\angle BAC=30°$ and $\angle ACB=90°$, place point D on side AC such that $\angle BDC=45°$. When $AD=2$, find the length x of BC.

[Sol] In $\triangle BCD$,

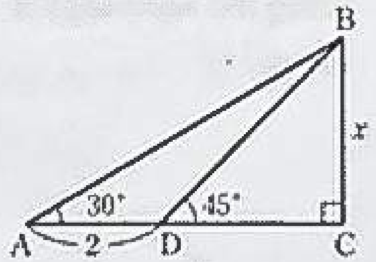
$BC=x$, $\angle BDC=45°$, $\angle BCD=90°$

$\therefore\ DC=x$

In $\triangle ABC$,

$$\frac{\sqrt{3}}{3}=\frac{x}{2+x} \quad \Longleftarrow \quad \boxed{\tan 30°=\frac{BC}{AD+DC}}$$

$\therefore\ x=\sqrt{3}+1$



3. Given right-angled triangle ABC with $\angle BAC=30°$ and $\angle ACB=90°$, place point D on side AC such that $\angle BDC=60°$. When $AD=4$, find the length x of BC.

[Sol] In $\triangle BCD$,

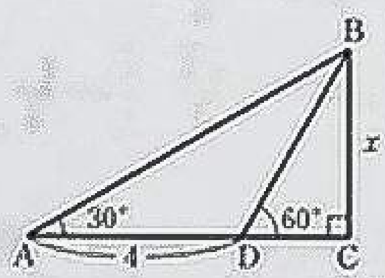
$BC=x$, $\angle BDC=60°$, $\angle BCD=90°$

$\therefore\ DC=\frac{\sqrt{3}}{3}x$

In $\triangle ABC$,

$$\frac{\sqrt{3}}{3}=\frac{x}{4+\frac{\sqrt{3}}{3}x}$$

$\therefore\ x=2\sqrt{3}$



4. Given right-angled triangle ABC with $\angle BAC=45°$ and $\angle ACB=90°$, place point D on side AC such that $\angle BDC=60°$. When $BC=1$, find the length x of AD.

[Sol] In $\triangle BCD$,

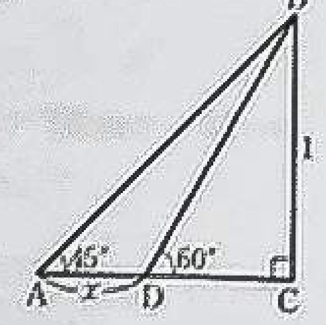
$BC=1$, $\angle BDC=60°$, $\angle BCD=90°$

$\therefore\ DC=\frac{\sqrt{3}}{3}$

In $\triangle ABC$,

$$1=\frac{1}{x+\frac{\sqrt{3}}{3}} \quad \Longleftarrow \quad \boxed{\tan 45°=\frac{BC}{AD+DC}}$$

$\therefore\ x=1-\frac{\sqrt{3}}{3}$



Trigonometric Ratios 1

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
mistakes 0	—	—	1	2~

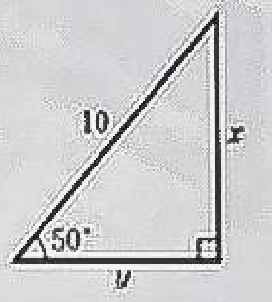
1. The following table shows the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ when $\theta=50°$, $60°$ and $70°$. Using the table, find the values of x and y of the following triangles. (Round off to one decimal place.)

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
50°	0.7660	0.6428	1.1918
60°	0.8660	0.5000	1.7321
70°	0.9397	0.3420	2.7475

※How to read the table for cos 60°

θ		$\cos\theta$
		↓
60°	→	0.5000

Ex.



[Sol] $\sin 50° = \dfrac{x}{10}$

$x = 10 \times 0.7660$ ← $\boxed{x=10\sin 50°}$

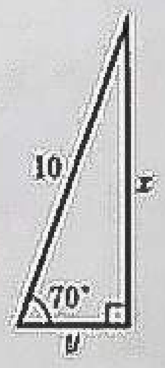
$= 7.66 \approx 7.7$

$\cos 50° = \dfrac{y}{10}$

$y = 10 \times 0.6428$

$= 6.428 \approx 6.4$

(1)



[Sol] $\sin 70° = \dfrac{x}{10}$

$x = 10 \times 0.9397$

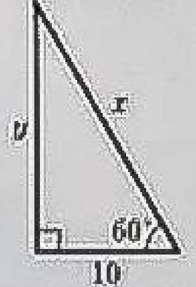
$= 9.397 \approx 9.4$

$\cos 70° = \dfrac{y}{10}$

$y = 10 \times 0.3420$

$= 3.42 \approx 3.4$

(2)



[Sol] $\cos 60° = \dfrac{10}{x}$

$x = \dfrac{10}{0.5000}$

$= 20$ $\quad [=20.0]$

$\tan 60° = \dfrac{y}{10}$

$y = 10 \times 1.7321$

$= 17.321 \approx 17.3$

Alternative Solution

$\sin 60° = \dfrac{y}{20}$

$y = 20 \times 0.8660$

$= 17.32 \approx 17.3$

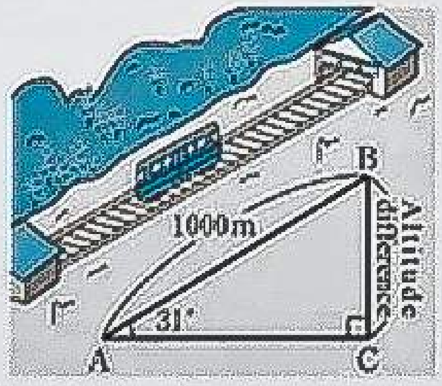
θ is read as "theta."

M89b

2. The table on the right shows the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for each degree from $\theta=25°$ to $34°$. Using the table, solve the following questions.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
25°	0.4226	0.9063	0.4663
26°	0.4384	0.8988	0.4877
27°	0.4540	0.8910	0.5095
28°	0.4695	0.8829	0.5317
29°	0.4848	0.8746	0.5543
30°	0.5000	0.8660	0.5774
31°	0.5150	0.8572	0.6009
32°	0.5299	0.8480	0.6249
33°	0.5446	0.8387	0.6494
34°	0.5592	0.8290	0.6745

(1) As shown on the right, a cable car runs on the route with the angle of elevation of 31°. When the distance between station A at the foot of the mountain and station B at the top is 1000m, find the altitude difference BC between stations A and B.



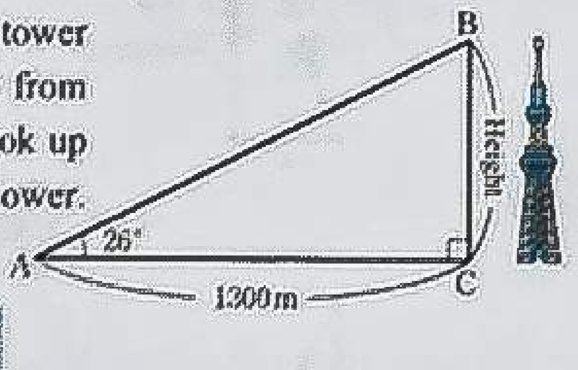
[Sol] $\sin 31° = \dfrac{BC}{1000}$ ⟵ $\sin 31° = \dfrac{BC}{AB}$

$\therefore BC = 1000 \times 0.5150$

$= 515$

Ans. 515 m

(2) When you look up at the top of a tower from point A which is 1300m away from the tower, the angle on which you look up is 26°. Find the height BC of the tower. (Round off to the nearest integer.)



[Sol] $\tan 26° = \dfrac{BC}{1300}$ ⟵ $\tan 26° = \dfrac{BC}{AC}$

$\therefore BC = 1300 \times 0.4877$

$= 634.01 \approx 634$

Ans. 634 m

Trigonometric Ratios 1

Name ______

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0			1	2

1. Given $\cos A=\frac{3}{4}$, find the values of $\sin A$ and $\tan A$. $(0^\circ < A < 90^\circ)$

[Sol] $\sin^2 A=1-\left(\frac{3}{4}\right)^2=\frac{7}{16}$

Since $\sin A>0$, $\sin A=\frac{\sqrt{7}}{4}$

Also, $\tan A=\frac{\sqrt{7}}{4}\div\frac{3}{4}=\frac{\sqrt{7}}{3}$

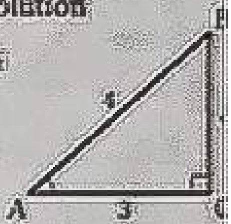
Alternative Solution

$BC=\sqrt{4^2-3^2}$

$=\sqrt{7}$

$\sin A=\frac{\sqrt{7}}{4}$

$\tan A=\frac{\sqrt{7}}{3}$



2. Given $\sin A=\frac{12}{13}$, find the values of $\cos A$ and $\tan A$. $(0^\circ < A < 90^\circ)$

[Sol] $\cos^2 A=1-\left(\frac{12}{13}\right)^2=\frac{25}{169}$

Since $\cos A>0$, $\cos A=\frac{5}{13}$

Also, $\tan A=\frac{12}{13}\div\frac{5}{13}=\frac{12}{5}$

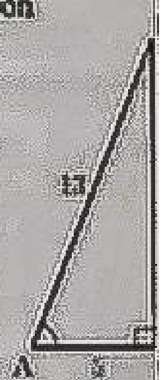
Alternative Solution

$AC=\sqrt{13^2-12^2}$

$=5$

$\cos A=\frac{5}{13}$

$\tan A=\frac{12}{5}$



3. Given $\tan A=\sqrt{2}$, find the values of $\cos A$ and $\sin A$. $(0^\circ < A < 90^\circ)$

➡ M87

[Sol] $\frac{1}{\cos^2 A}=1+(\sqrt{2})^2=3$

$\therefore\ \cos^2 A=\frac{1}{3}$

Since $\cos A>0$, $\cos A=\frac{\sqrt{3}}{3}$ $\left[=\frac{1}{\sqrt{3}}\right]$

Also, $\sin A=\sqrt{2}\cdot\frac{\sqrt{3}}{3}=\frac{\sqrt{6}}{3}$

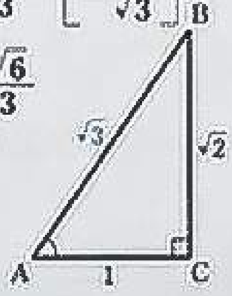
Alternative Solution

$AB=\sqrt{1^2+(\sqrt{2})^2}$

$=\sqrt{3}$

$\cos A=\frac{\sqrt{3}}{3}$ $\left[=\frac{1}{\sqrt{3}}\right]$

$\sin A=\frac{\sqrt{6}}{3}$



4. Given right-angled triangle ABC with ∠BAC=30° and ∠ACB=90°, place point D on side AC such that ∠BDC=45°. When BC=1, find the length x of AD.

➡ M88

[Sol] In △BCD,

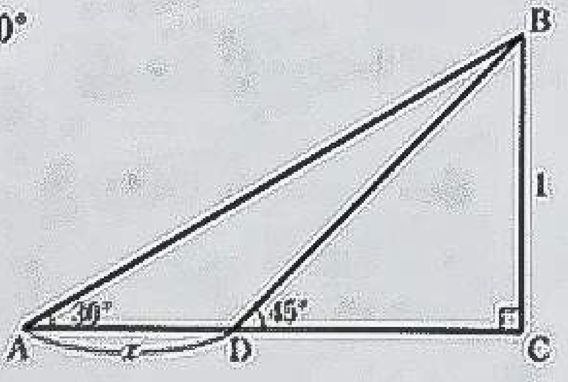
BC=1, ∠BDC=45°, ∠BCD=90°

$\therefore$ DC=1

In △ABC,

$\frac{\sqrt{3}}{3}=\frac{1}{x+1}$

$\therefore\ x=\sqrt{3}-1$

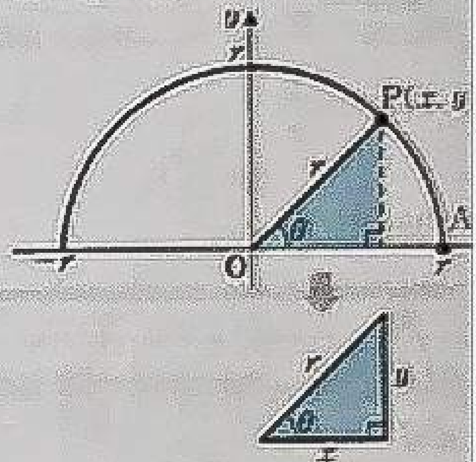


Trigonometric Ratios 2

Name
Date / /
Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0				

As shown in the diagram, draw a semicircle with the center at origin O and radius r, and let A be point $(r, 0)$. Place point $P(x, y)$ on the circumference of this semicircle and let $\angle AOP = \theta$.



Then, the following equations are true.

$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r},\ \tan\theta = \frac{y}{x}$$

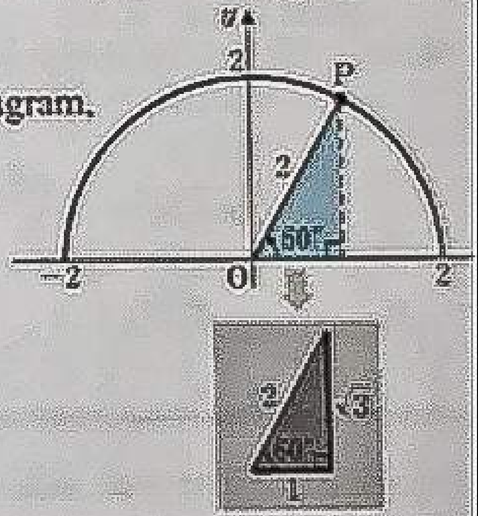
Ex. Using the semicircle with radius 2, find the values of $\sin 60^\circ$, $\cos 60^\circ$ a tan 60°.

[Sol] Placing point P as shown in the diagram, point P is $(1, \sqrt{3})$.

$$\therefore\ \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{1}{2}$$

$$\tan 60^\circ = \sqrt{3}$$



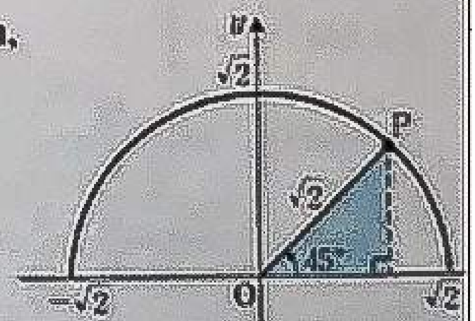
1. Using the semicircle with radius $\sqrt{2}$, find the values of $\sin 45^\circ$, cos 45 tan 45°.

[Sol] Placing point P as shown in the diagram, point P is $(1, 1)$.

$$\therefore\ \sin 45^\circ = \frac{\sqrt{2}}{2}\ \left[= \frac{1}{\sqrt{2}}\right]$$

$$\cos 45^\circ = \frac{\sqrt{2}}{2}\ \left[= \frac{1}{\sqrt{2}}\right]$$

$$\tan 45^\circ = 1$$



M91b

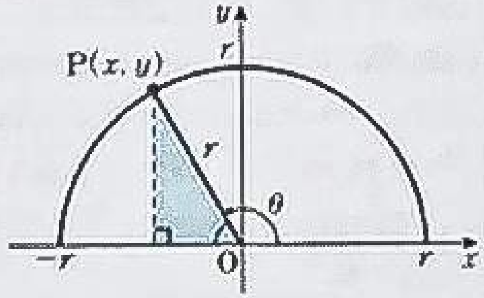
Given angle θ, where $0^\circ \leqq \theta \leqq 180^\circ$, $\sin\theta$, $\cos\theta$ and $\tan\theta$ are defined as follows.

Trigonometric Ratios of $0^\circ \leqq \theta \leqq 180^\circ$

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$



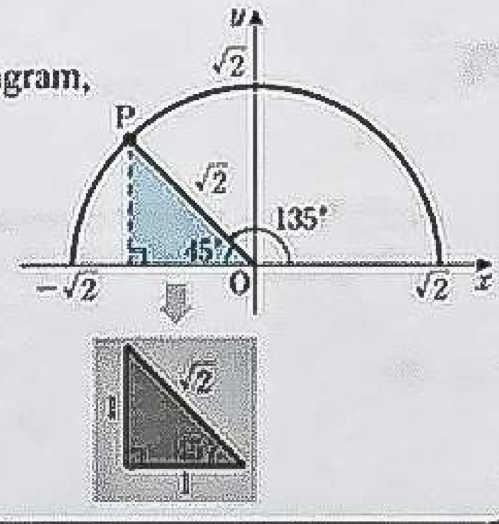
Ex Using the semicircle with radius $\sqrt{2}$, find the values of $\sin 135^\circ$, $\cos 135^\circ$ and $\tan 135^\circ$.

[Sol] Placing point P as shown in the diagram, point P is $(-1, 1)$.

$$\therefore\ \sin 135^\circ = \frac{\sqrt{2}}{2}$$

$$\cos 135^\circ = -\frac{\sqrt{2}}{2}$$

$$\tan 135^\circ = -1$$



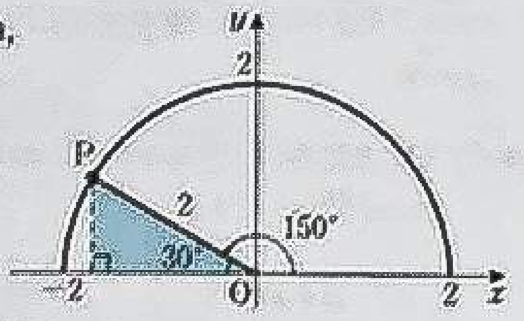
2. Using the semicircle with radius 2, find the values of $\sin 150^\circ$, $\cos 150^\circ$ and $\tan 150^\circ$.

[Sol] Placing point P as shown in the diagram, point P is $(-\sqrt{3}, 1)$.

$$\therefore\ \sin 150^\circ = \frac{1}{2}$$

$$\cos 150^\circ = -\frac{\sqrt{3}}{2}$$

$$\tan 150^\circ = -\frac{\sqrt{3}}{3}\ \left[= -\frac{1}{\sqrt{3}}\right]$$



M92a

KUMON

Trigonometric Ratios 2

Name

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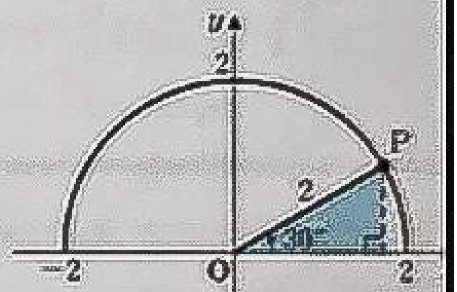
1. Using the semicircle with radius 2, find the values of $\sin 30^\circ$, $\cos 30^\circ$, $\tan 30^\circ$.



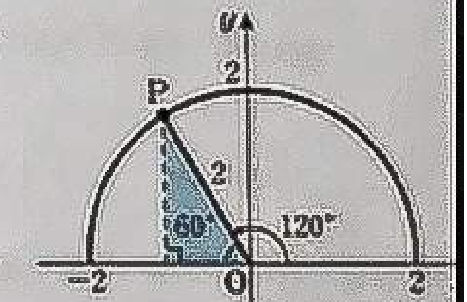
[Sol] Placing point P as shown in the diagram, point P is $(\sqrt{3}, 1)$.

$\therefore \sin 30^\circ = \frac{1}{2}$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$

$\tan 30^\circ = \frac{\sqrt{3}}{3} \quad \left[= \frac{1}{\sqrt{3}} \right]$

2. Using the semicircle with radius 2, find the values of $\sin 120^\circ$, $\cos 120^\circ$, $\tan 120^\circ$.



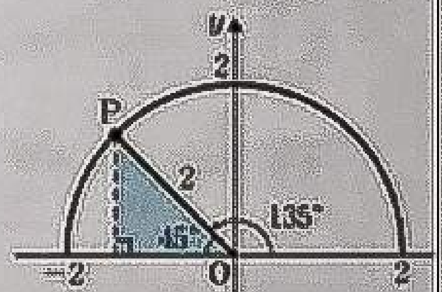
[Sol] Placing point P as shown in the diagram, point P is $(-1, \sqrt{3})$.

$\therefore \sin 120^\circ = \frac{\sqrt{3}}{2}$

$\cos 120^\circ = -\frac{1}{2}$

$\tan 120^\circ = -\sqrt{3}$

3. Using the semicircle with radius 2, find the values of $\sin 135^\circ$, $\cos 135^\circ$, $\tan 135^\circ$.



[Sol] Placing point P as shown in the diagram, point P is $(-\sqrt{2}, \sqrt{2})$.

$\therefore \sin 135^\circ = \frac{\sqrt{2}}{2}$

$\cos 135^\circ = -\frac{\sqrt{2}}{2}$

$\tan 135^\circ = -1$

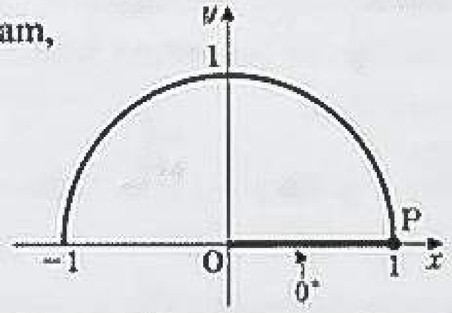
Ex Using the semicircle with radius 1, find the values of $\sin 0^\circ$, $\cos 0^\circ$ and $\tan 0^\circ$.

[Sol] Placing point P as shown in the diagram, point P is $(1, 0)$.

$\therefore\ \sin 0^\circ = 0$

$\cos 0^\circ = 1$

$\tan 0^\circ = 0$



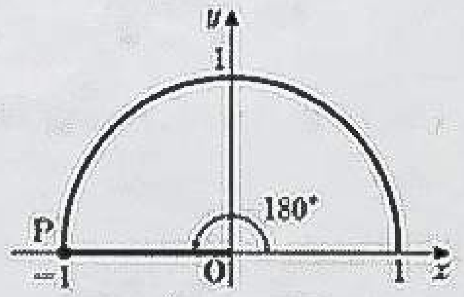
4. Using the semicircle with radius 1, find the values of $\sin 180^\circ$, $\cos 180^\circ$ and $\tan 180^\circ$.

[Sol] Placing point P as shown in the diagram, point P is $(-1, 0)$.

$\therefore\ \sin 180^\circ = 0$

$\cos 180^\circ = -1$

$\tan 180^\circ = 0$



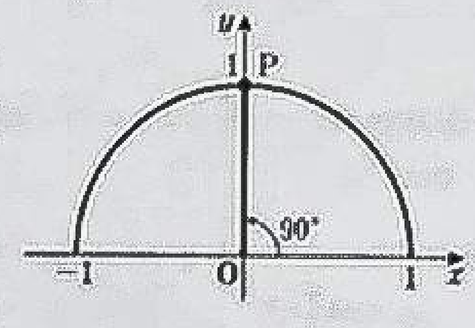
5. Using the semicircle with radius 1, find the values of $\sin 90^\circ$, $\cos 90^\circ$ and $\tan 90^\circ$.

[Sol] Placing point P as shown in the diagram, point P is ($\boxed{0}$, $\boxed{1}$).

$\therefore\ \sin 90^\circ = \boxed{1}$

$\cos 90^\circ = \boxed{0}$

The value of $\tan 90^\circ$ cannot be defined. (See Note)



Note Since $\tan\theta = \frac{y}{x}$, there are no values when $x = 0$.

Therefore, the value of $\tan 90^\circ$ "cannot be defined."

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Trigonometric Ratios 2

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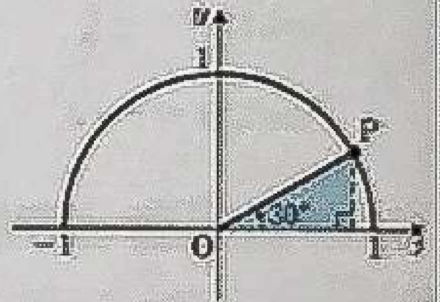
1. Using the semicircle with radius 1, find the values of $\sin 30^\circ$, $\cos 30^\circ$, $\tan 30^\circ$.

[Sol] Placing point P as shown in the diagram, point P is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

$\therefore\ \sin 30^\circ = \frac{1}{2}$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$

$\tan 30^\circ = \frac{\sqrt{3}}{3}\quad \left[= \frac{1}{\sqrt{3}}\right]$



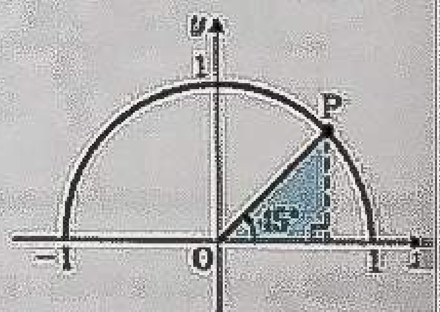
2. Using the semicircle with radius 1, find the values of $\sin 45^\circ$, $\cos 45^\circ$, $\tan 45^\circ$.

[Sol] Placing point P as shown in the diagram, point P is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

$\therefore\ \sin 45^\circ = \frac{\sqrt{2}}{2}\quad \left[= \frac{1}{\sqrt{2}}\right]$

$\cos 45^\circ = \frac{\sqrt{2}}{2}\quad \left[= \frac{1}{\sqrt{2}}\right]$

$\tan 45^\circ = 1$



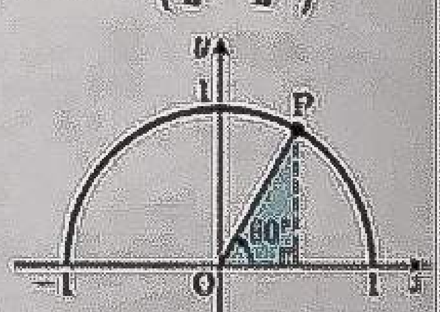
3. Using the semicircle with radius 1, find the values of $\sin 60^\circ$, $\cos 60^\circ$, $\tan 60^\circ$.

[Sol] Placing point P as shown in the diagram, point P is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$\therefore\ \sin 60^\circ = \frac{\sqrt{3}}{2}$

$\cos 60^\circ = \frac{1}{2}$

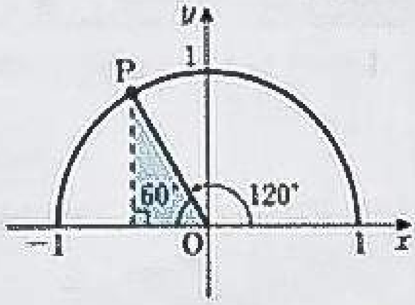
$\tan 60^\circ = \sqrt{3}$



M93b

4. Using the semicircle with radius 1, find the values of $\sin 120°$, $\cos 120°$ and $\tan 120°$.

[Sol] Placing point P as shown in the diagram,



point P is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

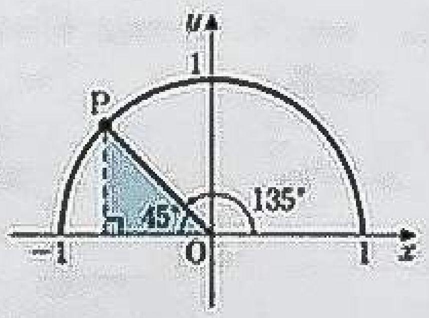
$\therefore\ \sin 120° = \frac{\sqrt{3}}{2}$

$\cos 120° = -\frac{1}{2}$

$\tan 120° = -\sqrt{3}$

5. Using the semicircle with radius 1, find the values of $\sin 135°$, $\cos 135°$ and $\tan 135°$.

[Sol] Placing point P as shown in the diagram,



point P is $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

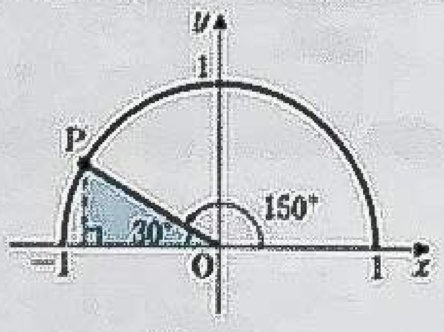
$\therefore\ \sin 135° = \frac{\sqrt{2}}{2}$ $\left[= \frac{1}{\sqrt{2}}\right]$

$\cos 135° = -\frac{\sqrt{2}}{2}$ $\left[= -\frac{1}{\sqrt{2}}\right]$

$\tan 135° = -1$

6. Using the semicircle with radius 1, find the values of $\sin 150°$, $\cos 150°$ and $\tan 150°$.

[Sol] Placing point P as shown in the diagram,



point P is $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

$\therefore\ \sin 150° = \frac{1}{2}$

$\cos 150° = -\frac{\sqrt{3}}{2}$

$\tan 150° = -\frac{\sqrt{3}}{3}$ $\left[= -\frac{1}{\sqrt{3}}\right]$

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Trigonometric Ratios 2

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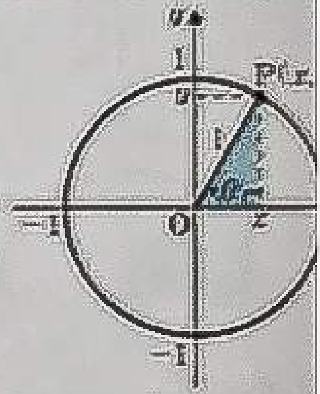
As shown in the diagram, place point P(x, y) on the circumference of a circle with radius 1.



Then, the following equations are true.

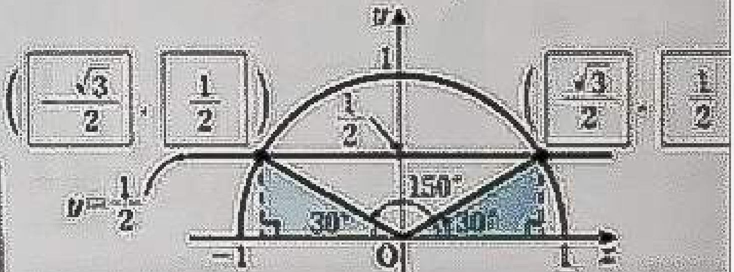
$$\sin\theta = y,\ \cos\theta = x,\ \tan\theta = \frac{y}{x}$$

Given $0° \leqq \theta \leqq 180°$, fill in the blanks in the diagrams and find the values which satisfy the following equations.

Ex. $\sin\theta = \frac{1}{2}$

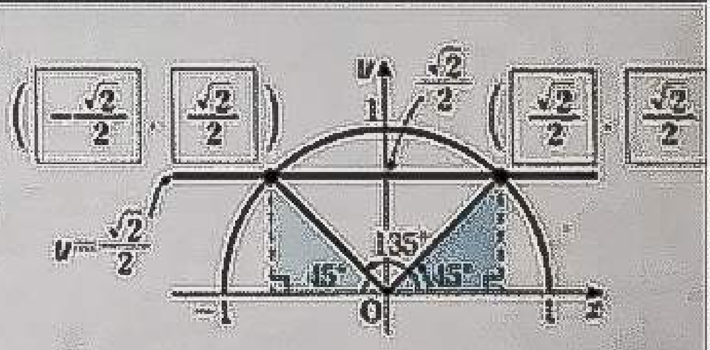
[Sol] $\theta = 30°, 150°$

The points of intersection of $y = \frac{1}{2}$ and the semicircle are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$



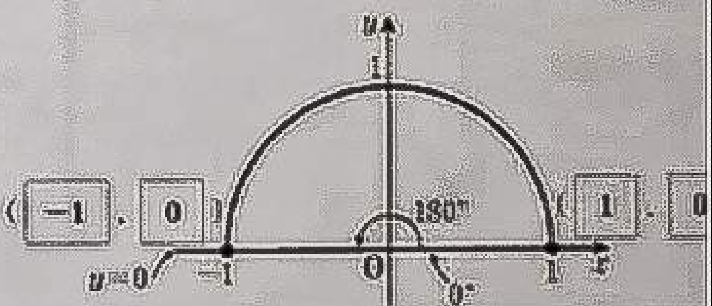
(1) $\sin\theta = \frac{\sqrt{2}}{2}$

[Sol] $\theta = 45°, 135°$



(2) $\sin\theta = 0$

[Sol] $\theta = 0°, 180°$



The circle with the center at the origin and radius 1 is called a ***unit circle***.

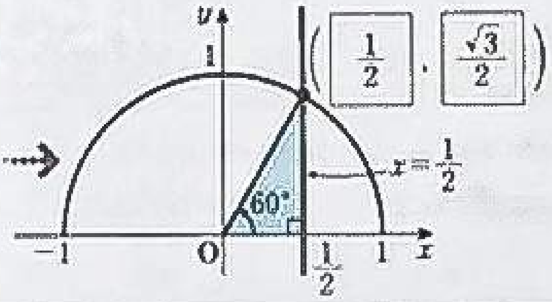
M94b

Ex.

$\cos\theta = \frac{1}{2}$

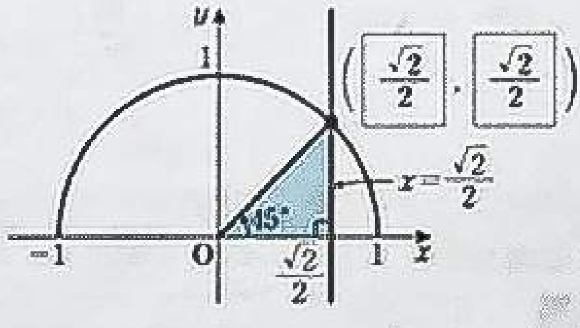
[Sol] $\theta = 60°$

The point of intersection of $x = \frac{1}{2}$ and the semicircle is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



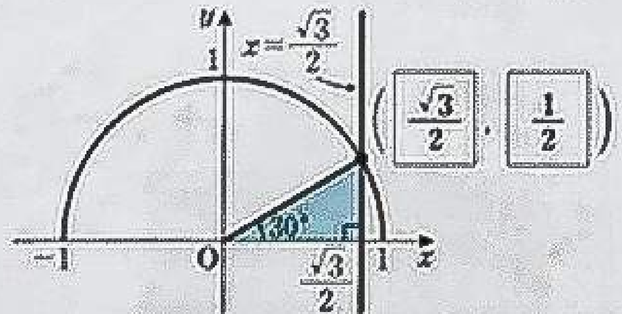
(3) $\cos\theta = \frac{\sqrt{2}}{2}$

[Sol] $\theta = 45°$



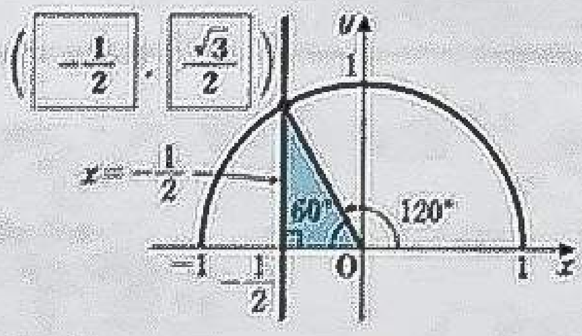
(4) $\cos\theta = \frac{\sqrt{3}}{2}$

[Sol] $\theta = 30°$



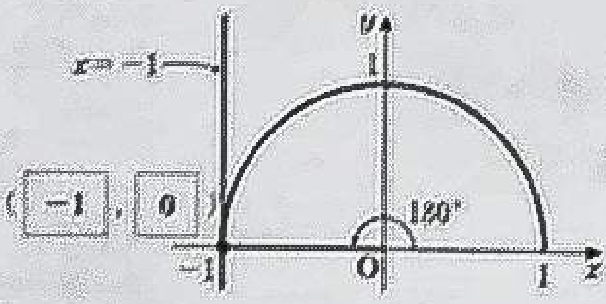
(5) $\cos\theta = -\frac{1}{2}$

[Sol] $\theta = 120°$



(6) $\cos\theta = -1$

[Sol] $\theta = 180°$



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Trigonometric Ratios 2

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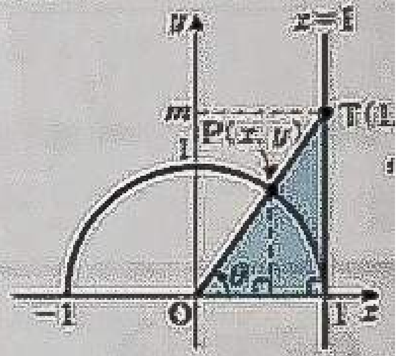
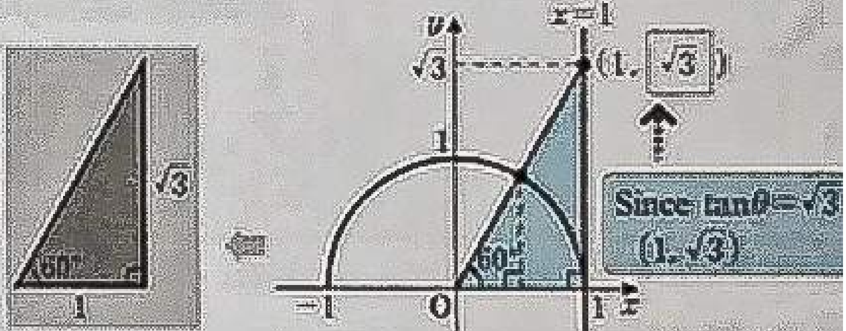
Given $\tan\theta = m$, place point $\mathrm{T}(1, m)$ on line $x=1$ as shown in the diagram.
Let $\mathrm{P}(x, y)$ be the point of intersection of line OT and the semicircle. Then, $m = \dfrac{m}{1} = \dfrac{y}{x} = \tan\theta$.



Given $0° \leqq \theta \leqq 180°$, fill in the blanks in the diagrams and find the values of
which satisfy the following equations.

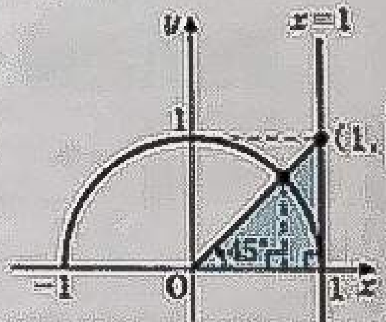
Ex. $\tan\theta = \sqrt{3}$

[Sol] $\theta = 60°$



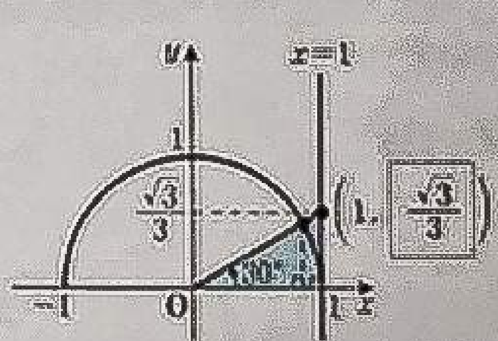
(1) $\tan\theta = 1$

[Sol] $\theta = 45°$



(2) $\tan\theta = \dfrac{\sqrt{3}}{3}$

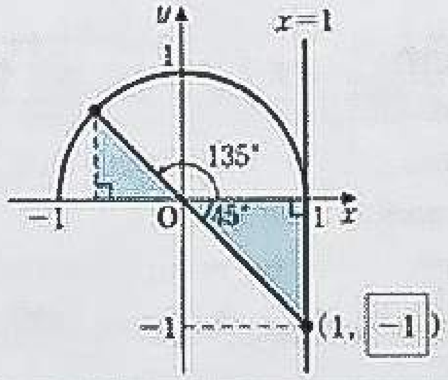
[Sol] $\theta = 30°$



M95b

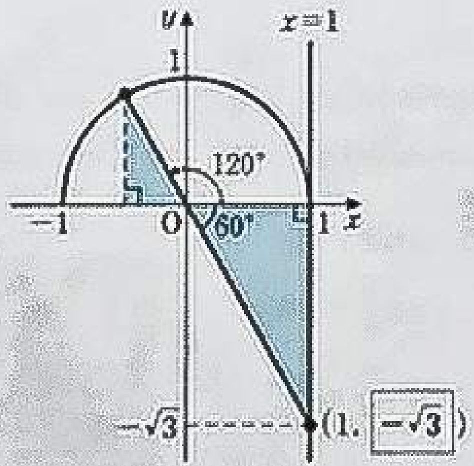
Ex $\tan\theta = -1$

[Sol] $\theta = 135°$



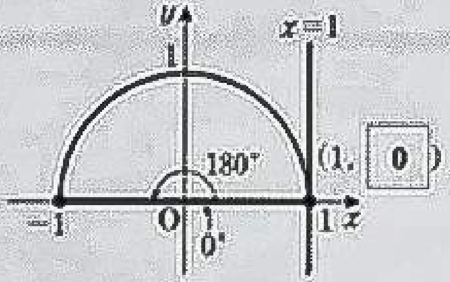
(3) $\tan\theta = -\sqrt{3}$

[Sol] $\theta = 120°$



(4) $\tan\theta = 0$

[Sol] $\theta = 0°, 180°$

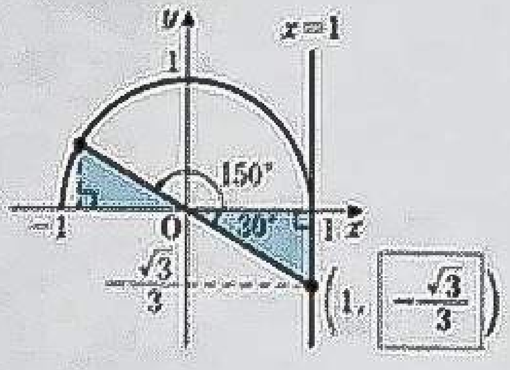


(5) $\sqrt{3}\tan\theta + 1 = 0$

[Sol] Rearranging,

$$\tan\theta = -\frac{\sqrt{3}}{3}$$

$$\theta = 150°$$



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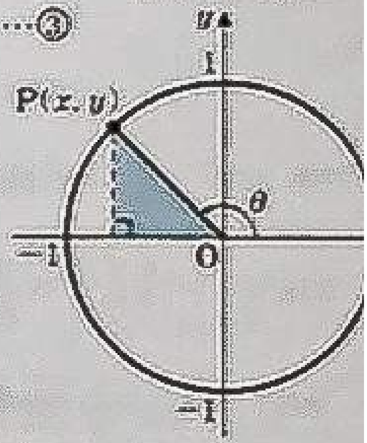
Fill in the following blanks.

[Sol] As shown in the diagram, placing point $P(x, y)$ on the circumference circle with radius 1,

$\sin\theta = y$ ···①, $\cos\theta = x$ ···②, $\tan\theta = \dfrac{y}{x}$ ···③

Substituting ① and ② into ③,

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$



Also, the length of OP in the diagram is 1.

Applying the Pythagorean Theorem,

$x^2 + y^2 = 1$ ···④

Substituting ① and ② into ④,

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

ers: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cos^2\theta + \sin^2\theta = 1$ $[\sin^2\theta + \cos^2\theta = 1]$

Ex. Given $\sin\theta = \dfrac{2}{3}$, find the values of $\cos\theta$ and $\tan\theta$. $(90° < \theta < 180°)$

[Sol] $\cos^2\theta = 1 - \left(\dfrac{2}{3}\right)^2 = \dfrac{5}{9}$ ⟵ $\boxed{\cos^2\theta = 1 - \sin^2\theta}$

Since $\cos\theta < 0$, $\cos\theta = -\dfrac{\sqrt{5}}{3}$ ⟵ When $90° < \theta < 180°$, $\cos\theta < 0$

Also, $\tan\theta = \dfrac{2}{3} \div \left(-\dfrac{\sqrt{5}}{3}\right) = -\dfrac{2\sqrt{5}}{5}$ ⟵ $\boxed{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$

1. Given $\sin\theta = \dfrac{3}{4}$, find the values of $\cos\theta$ and $\tan\theta$. $(90° < \theta < 180°)$

[Sol] $\cos^2\theta = 1 - \left(\dfrac{3}{4}\right)^2 = \dfrac{7}{16}$

Since $\cos\theta < 0$, $\cos\theta = -\dfrac{\sqrt{7}}{4}$

Also, $\tan\theta = \dfrac{3}{4} \div \left(-\dfrac{\sqrt{7}}{4}\right) = -\dfrac{3\sqrt{7}}{7}$ $\left[= -\dfrac{3}{\sqrt{7}}\right]$

M96b

2. Given $\cos\theta = -\frac{1}{4}$, find the values of $\sin\theta$ and $\tan\theta$. $(90° < \theta < 180°)$

[Sol] $\sin^2\theta = 1 - \left(-\frac{1}{4}\right)^2 = \frac{15}{16}$

Since $\sin\theta > 0$, $\sin\theta = \frac{\sqrt{15}}{4}$ ⟵ When $90° < \theta < 180°$, $\sin\theta > 0$

Also, $\tan\theta = \frac{\sqrt{15}}{4} \div \left(-\frac{1}{4}\right) = -\sqrt{15}$

3. Given $\sin\theta = \frac{1}{3}$, find the values of $\cos\theta$ and $\tan\theta$. $(0° \leqslant \theta \leqslant 90°)$

[Sol] $\cos^2\theta = 1 - \left(\frac{1}{3}\right)^2 = \frac{8}{9}$

Since $\cos\theta \geqslant 0$, $\cos\theta = \frac{2\sqrt{2}}{3}$ ⟵ When $0° \leqslant \theta \leqslant 90°$, $\cos\theta > 0$ or $\cos\theta = 0$

Also, $\tan\theta = \frac{1}{3} \div \frac{2\sqrt{2}}{3} = \frac{\sqrt{2}}{4}$ $\left[= \frac{1}{2\sqrt{2}}\right]$

4. Given $\sin\theta = \frac{3}{5}$, find the values of $\cos\theta$ and $\tan\theta$. $(0° \leqslant \theta \leqslant 180°)$

[Sol] $\cos^2\theta = 1 - \left(\frac{3}{5}\right)^2 = \frac{16}{25}$

When $0° \leqslant \theta \leqslant 90°$, $\cos\theta \geqslant 0$; therefore, $\cos\theta = \frac{4}{5}$

Also, $\tan\theta = \frac{3}{5} \div \frac{4}{5} = \frac{3}{4}$

When $90° < \theta \leqslant 180°$, $\cos\theta < 0$; therefore, $\cos\theta = -\frac{4}{5}$

Also, $\tan\theta = \frac{3}{5} \div \left(-\frac{4}{5}\right) = -\frac{3}{4}$

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Trigonometric Ratios 2

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Fill in the following blanks.

[Sol] Dividing both sides of the identity $\cos^2\theta+\sin^2\theta=1$ by $\cos^2\theta$,

$$1+\frac{\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}\quad\cdots①$$

Also, since $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ $\cdots$②, substituting ② into ①,

$$\boxed{1+\tan^2\theta=\frac{1}{\cos^2\theta}}$$

$1+\tan^2\theta=\dfrac{1}{\cos^2\theta}$

Ex. Given $\tan\theta=-2$, find the values of $\cos\theta$ and $\sin\theta$. $(0^\circ\leqq\theta\leqq180^\circ)$

[Sol] $\dfrac{1}{\cos^2\theta}=1+(-2)^2=5$ ⟵ $1+\tan^2\theta=\dfrac{1}{\cos^2\theta}$

$\therefore\ \cos^2\theta=\dfrac{1}{5}$

Since $\tan\theta<0$, $90^\circ<\theta<180^\circ$; therefore, $\cos\theta<0$

$\therefore\ \cos\theta=-\dfrac{\sqrt{5}}{5}$

$\therefore\ \sin\theta=-2\cdot\left(-\dfrac{\sqrt{5}}{5}\right)=\dfrac{2\sqrt{5}}{5}$ ⟵ Since $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$, $\sin\theta=\tan\theta\cos\theta$

1. Given $\tan\theta=-\dfrac{4}{3}$, find the values of $\cos\theta$ and $\sin\theta$. $(0^\circ\leqq\theta\leqq180^\circ)$

[Sol] $\dfrac{1}{\cos^2\theta}=1+\left(-\dfrac{4}{3}\right)^2=\dfrac{25}{9}$

$\therefore\ \cos^2\theta=\dfrac{9}{25}$

Since $\tan\theta<0$, $90^\circ<\theta<180^\circ$; therefore, $\cos\theta<0$

$\therefore\ \cos\theta=-\dfrac{3}{5}$

$\therefore\ \sin\theta=-\dfrac{4}{3}\cdot\left(-\dfrac{3}{5}\right)=\dfrac{4}{5}$

M97b

2. Given $\tan\theta = -\sqrt{5}$, find the values of $\cos\theta$ and $\sin\theta$. $(0^\circ \leqslant \theta \leqslant 180^\circ)$

[Sol] $\dfrac{1}{\cos^2\theta} = 1 + (-\sqrt{5})^2 = 6$

$\therefore\ \cos^2\theta = \dfrac{1}{6}$

Since $\tan\theta < 0$, $90^\circ < \theta < 180^\circ$; therefore, $\cos\theta < 0$

$\therefore\ \cos\theta = -\dfrac{\sqrt{6}}{6}\quad \left[= -\dfrac{1}{\sqrt{6}}\right]$

$\therefore\ \sin\theta = -\sqrt{5}\cdot\left(-\dfrac{\sqrt{6}}{6}\right) = \dfrac{\sqrt{30}}{6}$

3. Given $\tan\theta = \dfrac{1}{3}$, find the values of $\cos\theta$ and $\sin\theta$. $(0^\circ \leqslant \theta \leqslant 180^\circ)$

[Sol] $\dfrac{1}{\cos^2\theta} = 1 + \left(\dfrac{1}{3}\right)^2 = \dfrac{10}{9}$

$\therefore\ \cos^2\theta = \dfrac{9}{10}$

Since $\tan\theta > 0$, $0^\circ < \theta < 90^\circ$; therefore, $\cos\theta > 0$

$\therefore\ \cos\theta = \dfrac{3\sqrt{10}}{10}\quad \left[= \dfrac{3}{\sqrt{10}}\right]$

$\therefore\ \sin\theta = \dfrac{1}{3}\cdot\dfrac{3\sqrt{10}}{10} = \dfrac{\sqrt{10}}{10}\quad \left[= \dfrac{1}{\sqrt{10}}\right]$

Trigonometric Ratios 2

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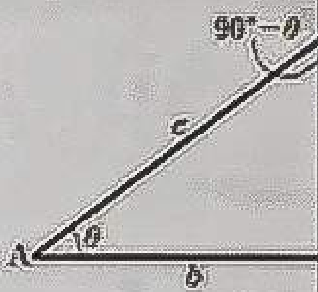
Given the right-angled triangle in the diagram, let one of the angles other the right angle be θ. The other angle is thus expressed as $90°-\theta$. Expres following trigonometric ratios using a, b and c.



[1] $\sin\theta = \boxed{\frac{a}{c}}$ [4] $\sin(90°-\theta) = \boxed{\frac{b}{c}}$

[2] $\cos\theta = \boxed{\frac{b}{c}}$ [5] $\cos(90°-\theta) = \boxed{\frac{a}{c}}$

[3] $\tan\theta = \boxed{\frac{a}{b}}$ [6] $\tan(90°-\theta) = \boxed{\frac{b}{a}}$

[1] $\frac{a}{c}$ [2] $\frac{b}{c}$ [3] $\frac{a}{b}$ [4] $\frac{b}{c}$ [5] $\frac{a}{c}$ [6] $\frac{b}{a}$

From [2] and [4], $\cos\theta = \sin(90°-\theta)$

From [1] and [5], $\sin\theta = \cos(90°-\theta)$

From [3] and [6], $\tan\theta = \dfrac{1}{\tan(90°-\theta)}$

Trigonometric Ratios of $90°-\theta$

$\sin(90°-\theta) = \cos\theta$

$\cos(90°-\theta) = \sin\theta$

$\tan(90°-\theta) = \dfrac{1}{\tan\theta}$

1. Using the formulas above, fill in the following blanks.

(1) $\sin 50° = \cos \boxed{40}°$ (4) $\sin 54° = \cos \boxed{36}°$

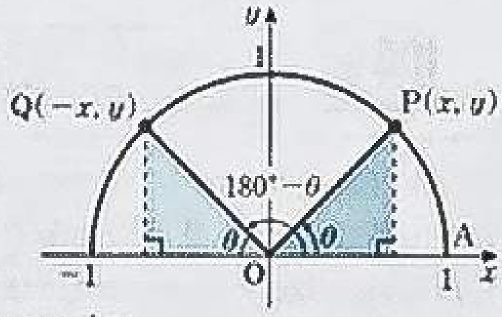
(2) $\cos 70° = \sin \boxed{20}°$ (5) $\cos 82° = \sin \boxed{8}°$

(3) $\tan 80° = \dfrac{1}{\tan \boxed{10}°}$ (6) $\tan 53° = \dfrac{1}{\tan \boxed{37}°}$

For the angles of $\triangle ABC$, $\angle A + \angle B + \angle C = 180°$ is true.
As shown in the diagram above, when $\angle A = \theta$ and $\angle C = 90°$, then $\angle B = 90° - \theta$.

M98b

As shown in the diagram, place points P and Q on the circumference of a semicircle with radius 1 where $\angle AOP=\theta$ and $\angle AOQ=180^\circ-\theta$. Let point P be $P(x, y)$. Since point Q is symmetric to point P with respect to the y-axis, $Q(-x, y)$ is true.



Express the following trigonometric ratios using x and y.

[1] $\sin\theta=\boxed{y}$

[2] $\cos\theta=\boxed{x}$

[3] $\tan\theta=\boxed{\dfrac{y}{x}}$

[4] $\sin(180^\circ-\theta)=\boxed{y}$

[5] $\cos(180^\circ-\theta)=\boxed{-x}$

[6] $\tan(180^\circ-\theta)=\boxed{-\dfrac{y}{x}}$

Answers: [1] y, [2] x, [3] $\dfrac{y}{x}$, [4] y, [5] $-x$, [6] $-\dfrac{y}{x}$

From [1] and [4], $\sin\theta=\sin(180^\circ-\theta)$

From [2] and [5], $\cos\theta=-\cos(180^\circ-\theta)$

From [3] and [6], $\tan\theta=-\tan(180^\circ-\theta)$

Trigonometric Ratios of $180^\circ-\theta$

$\sin(180^\circ-\theta)=\sin\theta$

$\cos(180^\circ-\theta)=-\cos\theta$

$\tan(180^\circ-\theta)=-\tan\theta$

2. Using the formulas above, fill in the following blanks.

(1) $\sin 115^\circ=\sin\boxed{65}^\circ$

(2) $\cos 115^\circ=-\cos\boxed{65}^\circ$

(3) $\tan 115^\circ=-\tan\boxed{65}^\circ$

(4) $\sin 145^\circ=\sin\boxed{35}^\circ$

(5) $\cos 105^\circ=-\cos\boxed{75}^\circ$

(6) $\tan 158^\circ=-\tan\boxed{22}^\circ$

M99a

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Trigonometric Ratios 2

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Time : to :

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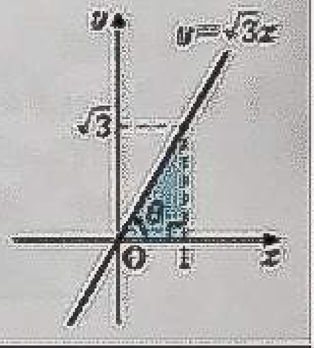
1. Find the angle θ formed by each given line and the positive x-axis. $(0° \leqq \theta \leqq 180°)$

Ex. $y=\sqrt{3}x$

[Sol] Since $\tan\theta=\sqrt{3}$,

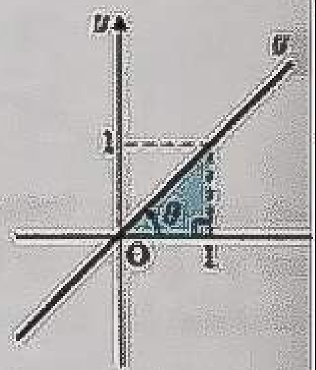
$\theta=60°$



(1) $y=x$

[Sol] Since $\tan\theta=1$,

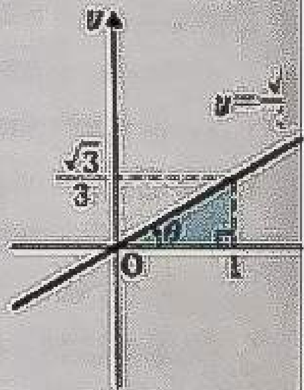
$\theta=45°$



(2) $y=\frac{\sqrt{3}}{3}x$

[Sol] Since $\tan\theta=\frac{\sqrt{3}}{3}$,

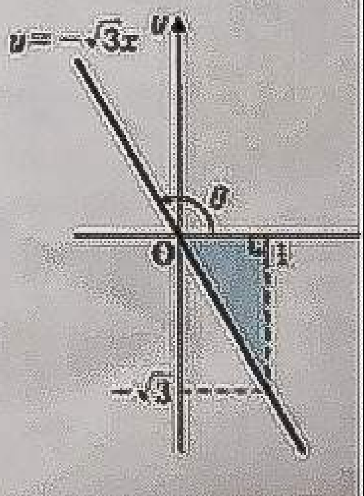
$\theta=30°$



(3) $y=-\sqrt{3}x$

[Sol] Since $\tan\theta=-\sqrt{3}$,

$\theta=120°$



M99b

2. Let the angles formed by the positive x-axis and two lines $y=\frac{\sqrt{3}}{3}x$ …① and $y=\sqrt{3}x$ …② be α and β respectively. Find the angle θ formed by these two lines. $(0° \leqslant \theta \leqslant 90°)$

[Sol] From ①, $\tan\alpha=\frac{\sqrt{3}}{3}$

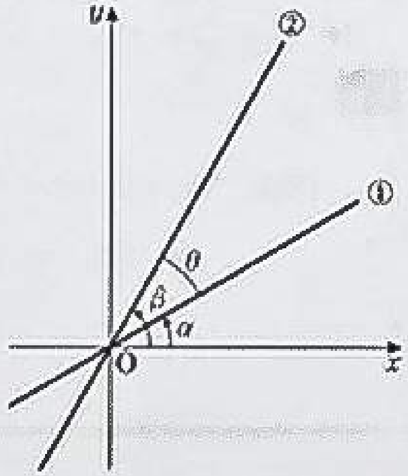
$\therefore\ \alpha=30°$

From ②, $\tan\beta=\sqrt{3}$

$\therefore\ \beta=60°$

$\therefore\ \theta=60°-30°$ ⟵ $\boxed{\theta=\beta-\alpha}$

$=30°$



3. Let the angles formed by the positive x-axis and two lines $\sqrt{3}x-y=0$ …① and $x+y=0$ …② be α and β respectively. Find the angle θ formed by these two lines. $(0° \leqslant \theta \leqslant 90°)$

[Sol] From ①, $y=\sqrt{3}x$

$\therefore\ \tan\alpha=\sqrt{3}$

$\therefore\ \alpha=60°$

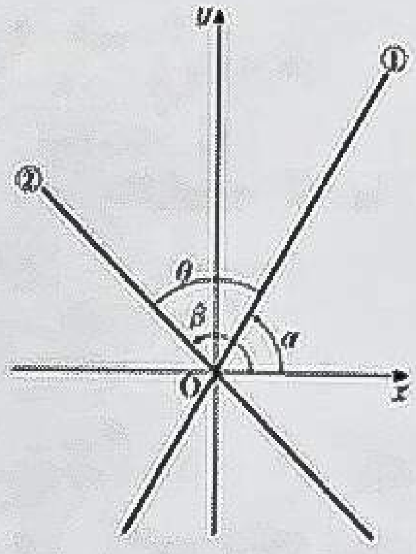
From ②, $y=-x$

$\therefore\ \tan\beta=-1$

$\therefore\ \beta=125°$

$\therefore\ \theta=125°-60°$

$=75°$



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Trigonometric Ratios 2

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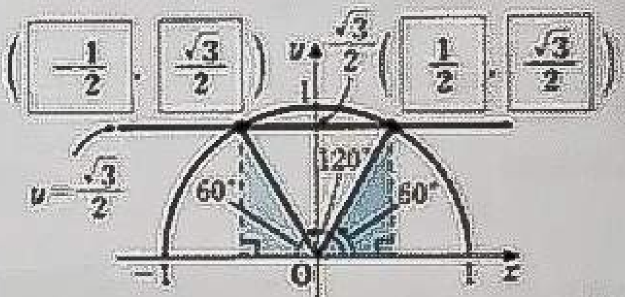
1. Given $0^\circ \leqq \theta \leqq 180^\circ$, fill in the blanks in the diagrams and find the values of θ which satisfy the following equations.

➡ M94

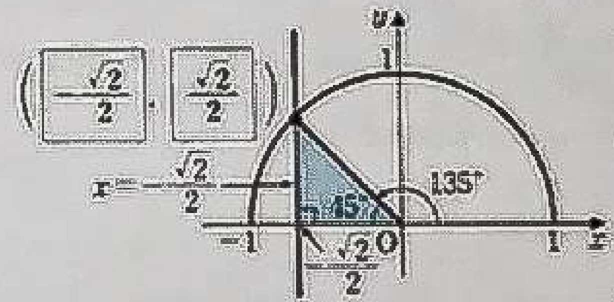
(1) $\sin\theta = \frac{\sqrt{3}}{2}$

[Sol] $\theta = 60^\circ, 120^\circ$



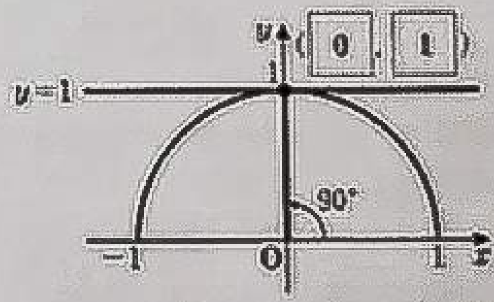
(2) $\cos\theta = -\frac{\sqrt{2}}{2}$

[Sol] $\theta = 135^\circ$



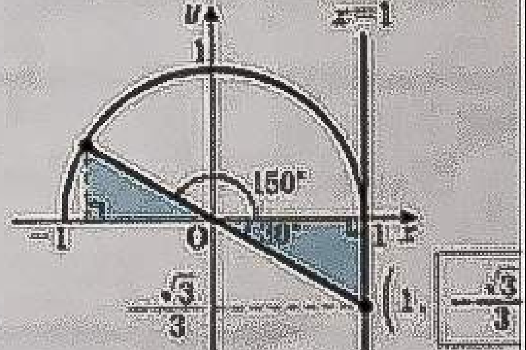
(3) $\sin\theta = 1$

[Sol] $\theta = 90^\circ$



(4) $\tan\theta = -\frac{\sqrt{3}}{3}$

[Sol] $\theta = 150^\circ$



2. Given $\cos\theta = -\frac{\sqrt{2}}{2}$, find the values of $\sin\theta$ and $\tan\theta$. $(90° < \theta < 180°)$

➡ M96

[Sol] $\sin^2\theta = 1 - \left(-\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$

Since $\sin\theta > 0$,

$\sin\theta = \frac{\sqrt{2}}{2}$ $\left[= \frac{1}{\sqrt{2}}\right]$

Also, $\tan\theta = \frac{\sqrt{2}}{2} \div \left(-\frac{\sqrt{2}}{2}\right) = -1$

[Alternative Solution

$\tan^2\theta = \frac{1}{\left(-\frac{\sqrt{2}}{2}\right)^2} - 1 = 1$

Since $\tan\theta < 0$, $\tan\theta = -1$

Also, $\sin\theta = -1 \cdot \left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$]

3. Given $\sin\theta = \frac{\sqrt{3}}{2}$, find the values of $\cos\theta$ and $\tan\theta$. $(90° < \theta < 180°)$

➡ M96

[Sol] $\cos^2\theta = 1 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4}$

Since $\cos\theta < 0$, $\cos\theta = -\frac{1}{2}$

Also, $\tan\theta = \frac{\sqrt{3}}{2} \div \left(-\frac{1}{2}\right) = -\sqrt{3}$

4. Given $\tan\theta = -\sqrt{2}$, find the values of $\cos\theta$ and $\sin\theta$. $(0° \leqq \theta \leqq 180°)$

➡ M97

[Sol] $\frac{1}{\cos^2\theta} = 1 + (-\sqrt{2})^2 = 3$

$\therefore \cos^2\theta = \frac{1}{3}$

Since $\tan\theta < 0$, $90° < \theta < 180°$; therefore $\cos\theta < 0$

$\therefore \cos\theta = -\frac{\sqrt{3}}{3}$ $\left[= -\frac{1}{\sqrt{3}}\right]$

$\therefore \sin\theta = -\sqrt{2} \cdot \left(-\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{6}}{3}$

Properties of Trigonometric Functions 1

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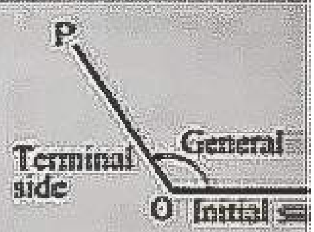
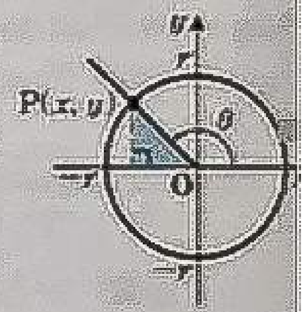
When OP with point O as the center is rotated on a plane, OP is called a ***terminal side*** and its original position OX is called an ***initial side***. An angle obtained by rotating OP from the initial side about point O is called a ***general angle***.



Assume that the positive x-axis is the initial side on a coordinate plane. Let P(x, … be the coordinates of the point, such that the terminal side of general ang… intersects at the circle with radius r. Then, the following are defined.

Definitions of Trigonometric Functions

$$\sin\theta=\frac{y}{r},\quad \cos\theta=\frac{x}{r},\quad \tan\theta=\frac{y}{x}$$



Also, when the length of radius r of the circle is 1,

$$\sin\theta=y,\quad \cos\theta=x,\quad \tan\theta=\frac{y}{x}$$

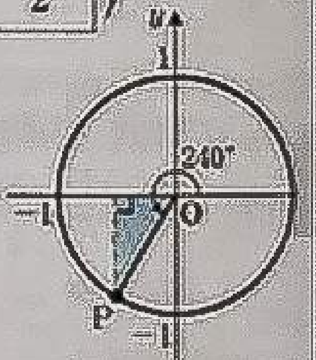
Ex. Place point P on the circumference of a circle with radius 1 such th… general angle $\theta=240^\circ$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\boxed{-\frac{1}{2}}, \boxed{-\frac{\sqrt{3}}{2}}\right)$.

$$\therefore\ \sin\theta=\boxed{-\frac{\sqrt{3}}{2}}$$

$$\cos\theta=\boxed{-\frac{1}{2}}$$

$$\tan\theta=\boxed{\sqrt{3}}$$



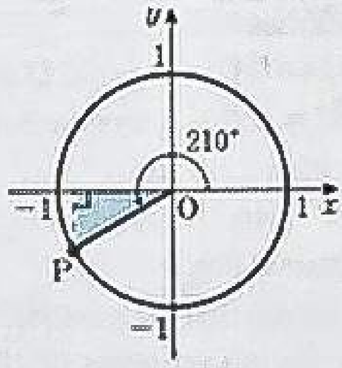
Ans: $-\frac{1}{2}$, $-\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}$, $-\frac{1}{2}$, $\sqrt{3}$

When considering the general angle on a coordinate plane, the initial side is generally p… on the positive x-axis.

M101b

1. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=210°$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.



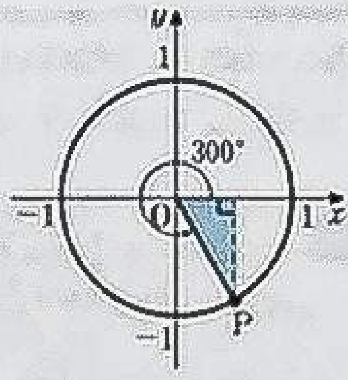
$\therefore\ \sin\theta=-\frac{1}{2}$

$\cos\theta=-\frac{\sqrt{3}}{2}$

$\tan\theta=\frac{\sqrt{3}}{3}$

2. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=300°$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.



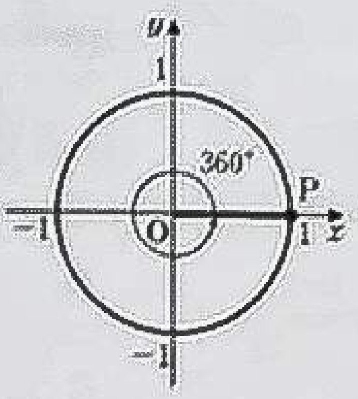
$\therefore\ \sin\theta=-\frac{\sqrt{3}}{2}$

$\cos\theta=\frac{1}{2}$

$\tan\theta=-\sqrt{3}$

3. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=360°$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $(1, 0)$.



$\therefore\ \sin\theta=0$

$\cos\theta=1$

$\tan\theta=0$

Properties of Trigonometric Functions 1

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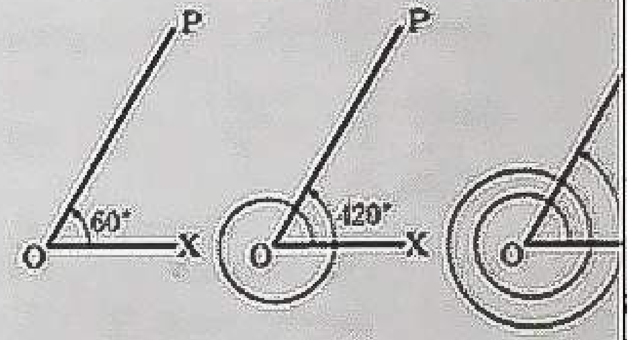
Rotating 360°, the terminal side goes back to the original position. Therefore shown in the diagram, the terminal side OP of 60° and the terminal sides of the following angles are congruent:

$420° = 60° + 360° \times 1$

$780° = 60° + 360° \times 2$



The general angle θ represented by terminal side OP is expressed as follows:

$\theta = \alpha + 360° \times n$ (n is an integer)

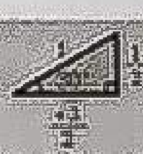
Ex. Place point P on the circumference of a circle with radius 1 such th general angle $\theta = 390°$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

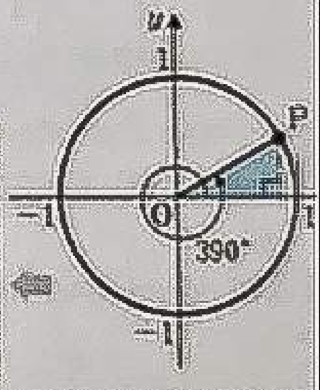
$\therefore \sin\theta = \frac{1}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\tan\theta = \frac{\sqrt{3}}{3}$



$\theta = 390°$ and $\theta = 30°$ have the same values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.



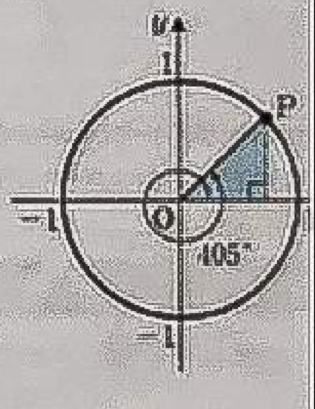
1. Place point P on the circumference of a circle with radius 1 such that ger angle $\theta = 405°$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

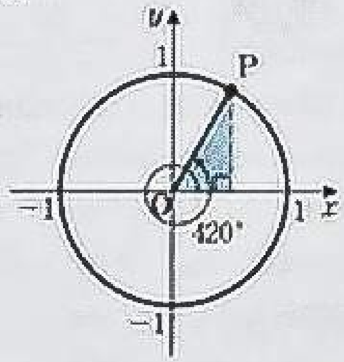
$\therefore \sin\theta = \frac{\sqrt{2}}{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\tan\theta = 1$



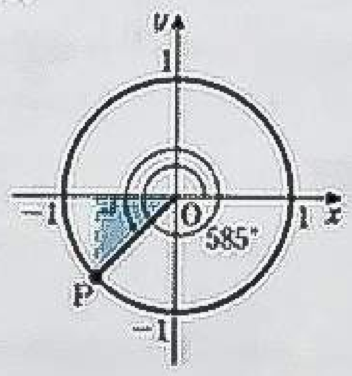
2. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=420^\circ$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.



[Sol] The coordinates of point P are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$$\therefore \sin\theta=\frac{\sqrt{3}}{2}$$
$$\cos\theta=\frac{1}{2}$$
$$\tan\theta=\sqrt{3}$$

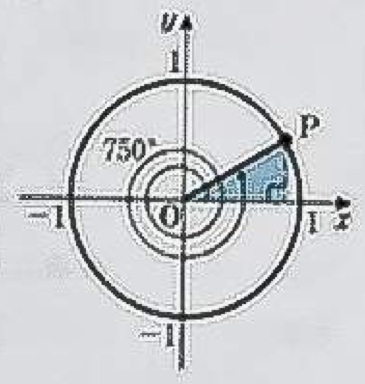
3. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=585^\circ$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.



[Sol] The coordinates of point P are $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

$$\therefore \sin\theta=-\frac{\sqrt{2}}{2}$$
$$\cos\theta=-\frac{\sqrt{2}}{2}$$
$$\tan\theta=1$$

4. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=750^\circ$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.



[Sol] The coordinates of point P are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

$$\therefore \sin\theta=\frac{1}{2}$$
$$\cos\theta=\frac{\sqrt{3}}{2}$$
$$\tan\theta=\frac{\sqrt{3}}{3}$$

The relationships between trigonometric functions of $\theta+360^\circ\times n$ (where n is an integer) and θ are as follows.

Trigonometric Functions of $\theta+360^\circ\times n$

$\sin(\theta+360^\circ\times n)=\sin\theta$

$\cos(\theta+360^\circ\times n)=\cos\theta$

$\tan(\theta+360^\circ\times n)=\tan\theta$

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Properties of Trigonometric Functions 1

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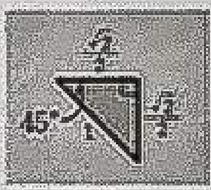
Ex. Place point P on the circumference of a circle with radius 1 such tha
general angle $\theta = -45^\circ$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

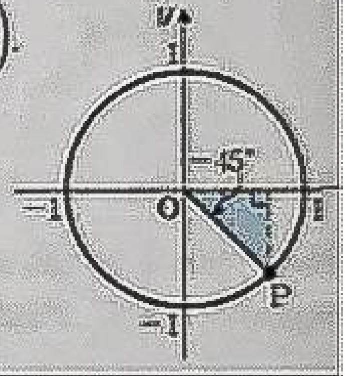
[Sol] The coordinates of point P are $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

$\therefore \sin\theta = -\frac{\sqrt{2}}{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\tan\theta = -1$





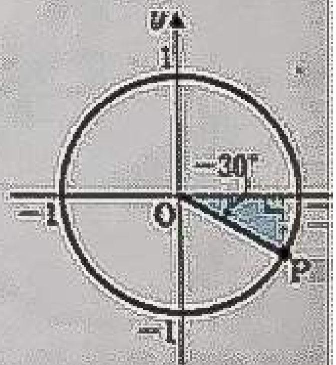
1. Place point P on the circumference of a circle with radius 1 such that gen
angle $\theta = -30^\circ$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

$\therefore \sin\theta = -\frac{1}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\tan\theta = -\frac{\sqrt{3}}{3}$



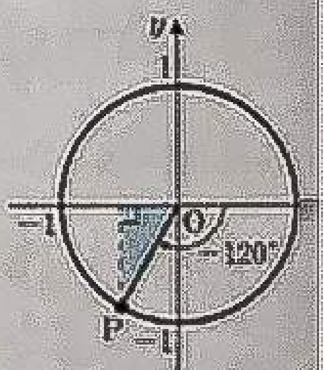
2. Place point P on the circumference of a circle with radius 1 such that gen
angle $\theta = -120^\circ$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

$\therefore \sin\theta = -\frac{\sqrt{3}}{2}$

$\cos\theta = -\frac{1}{2}$

$\tan\theta = \sqrt{3}$



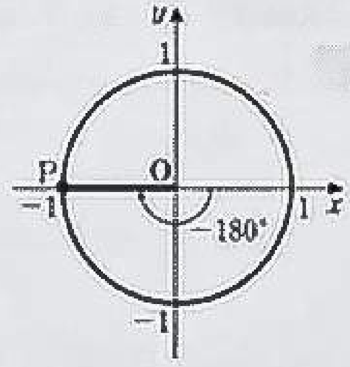
3. Place point P on the circumference of a circle with radius 1 such that general angle $\theta = -180^\circ$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $(-1, 0)$.

$\therefore\ \sin\theta = 0$

$\cos\theta = -1$

$\tan\theta = 0$



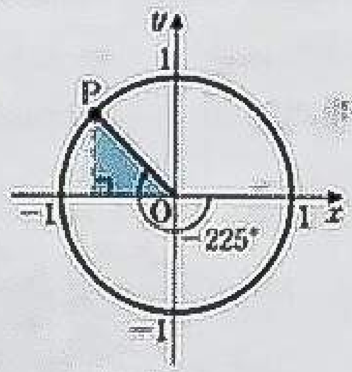
4. Place point P on the circumference of a circle with radius 1 such that general angle $\theta = -225^\circ$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

$\therefore\ \sin\theta = \dfrac{\sqrt{2}}{2}$

$\cos\theta = -\dfrac{\sqrt{2}}{2}$

$\tan\theta = -1$



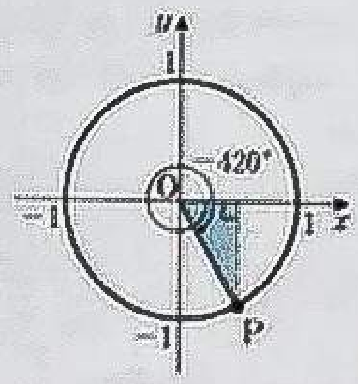
5. Place point P on the circumference of a circle with radius 1 such that general angle $\theta = -420^\circ$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$.

$\therefore\ \sin\theta = -\dfrac{\sqrt{3}}{2}$

$\cos\theta = \dfrac{1}{2}$

$\tan\theta = -\sqrt{3}$



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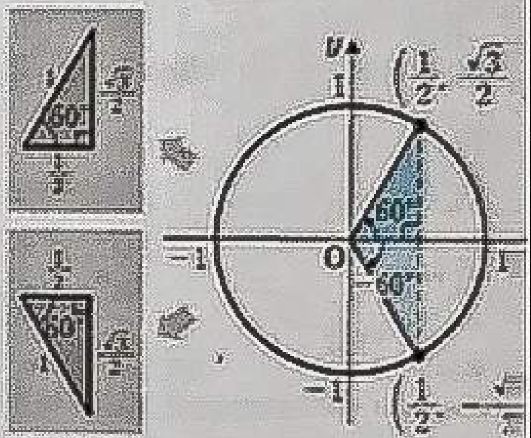
For each given θ, find the values of sin, cos and tan of θ and $-\theta$.

Ex. $\theta=60^\circ$

[Sol] $\sin\theta=\frac{\sqrt{3}}{2}$ $\quad\sin(-\theta)=-\frac{\sqrt{3}}{2}$

$\cos\theta=\frac{1}{2}$ $\quad\cos(-\theta)=\frac{1}{2}$

$\tan\theta=\sqrt{3}$ $\quad\tan(-\theta)=-\sqrt{3}$

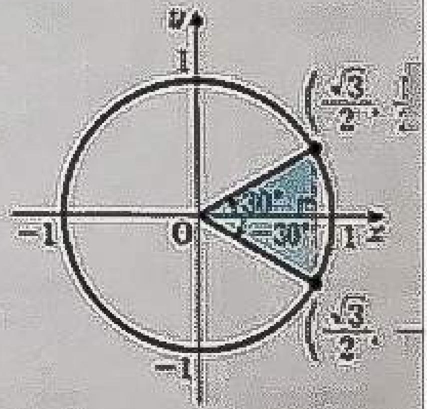


(1) $\theta=30^\circ$

[Sol] $\sin\theta=\frac{1}{2}$ $\quad\sin(-\theta)=-\frac{1}{2}$

$\cos\theta=\frac{\sqrt{3}}{2}$ $\quad\cos(-\theta)=\frac{\sqrt{3}}{2}$

$\tan\theta=\frac{\sqrt{3}}{3}$ $\quad\tan(-\theta)=-\frac{\sqrt{3}}{3}$

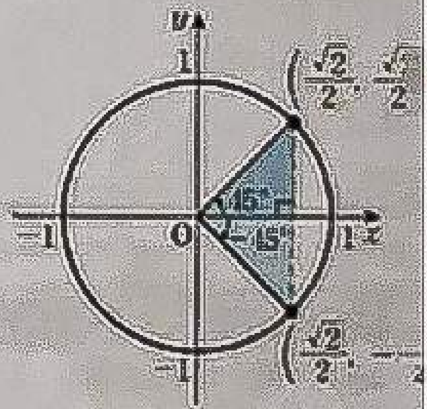


(2) $\theta=45^\circ$

[Sol] $\sin\theta=\frac{\sqrt{2}}{2}$ $\quad\sin(-\theta)=-\frac{\sqrt{2}}{2}$

$\cos\theta=\frac{\sqrt{2}}{2}$ $\quad\cos(-\theta)=\frac{\sqrt{2}}{2}$

$\tan\theta=1$ $\quad\tan(-\theta)=-1$



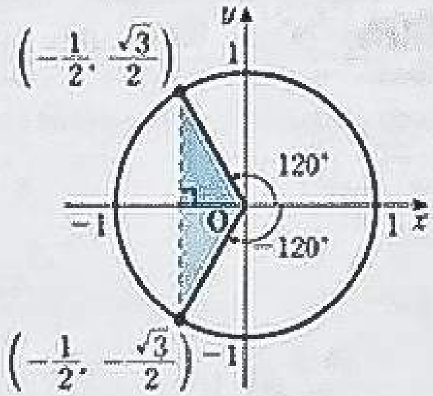
M104b

(3) $\theta=120^\circ$

[Sol] $\sin\theta=\dfrac{\sqrt{3}}{2}$ $\sin(-\theta)=-\dfrac{\sqrt{3}}{2}$

$\cos\theta=-\dfrac{1}{2}$ $\cos(-\theta)=-\dfrac{1}{2}$

$\tan\theta=-\sqrt{3}$ $\tan(-\theta)=\sqrt{3}$

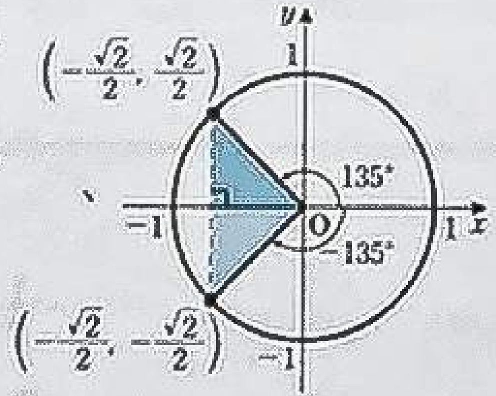


(4) $\theta=135^\circ$

[Sol] $\sin\theta=\dfrac{\sqrt{2}}{2}$ $\sin(-\theta)=-\dfrac{\sqrt{2}}{2}$

$\cos\theta=-\dfrac{\sqrt{2}}{2}$ $\cos(-\theta)=-\dfrac{\sqrt{2}}{2}$

$\tan\theta=-1$ $\tan(-\theta)=1$

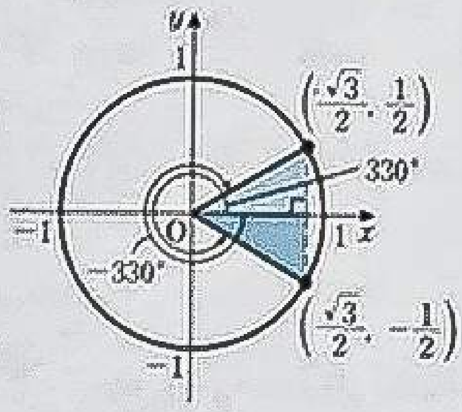


(5) $\theta=330^\circ$

[Sol] $\sin\theta=-\dfrac{1}{2}$ $\sin(-\theta)=\dfrac{1}{2}$

$\cos\theta=\dfrac{\sqrt{3}}{2}$ $\cos(-\theta)=\dfrac{\sqrt{3}}{2}$

$\tan\theta=-\dfrac{\sqrt{3}}{3}$ $\tan(-\theta)=\dfrac{\sqrt{3}}{3}$



The relationships between trigonometric functions of $-\theta$ and θ are as follows.

Trigonometric Functions of $-\theta$

$\sin(-\theta)=-\sin\theta$

$\cos(-\theta)=\cos\theta$

$\tan(-\theta)=-\tan\theta$

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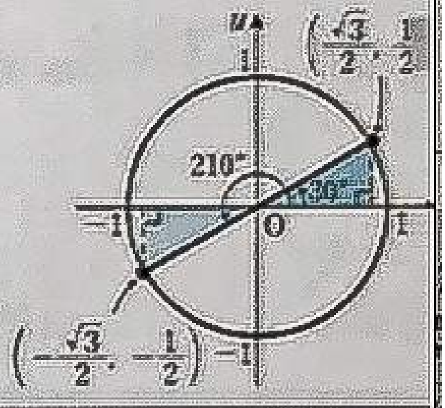
For each given θ, find the values of sin, cos and tan of θ and $\theta + 180°$.

Ex. $\theta = 30°$

[Sol] $\sin\theta = \dfrac{1}{2}$ $\quad \sin(\theta + 180°) = -\dfrac{1}{2}$

$\cos\theta = \dfrac{\sqrt{3}}{2}$ $\quad \cos(\theta + 180°) = -\dfrac{\sqrt{3}}{2}$

$\tan\theta = \dfrac{\sqrt{3}}{3}$ $\quad \tan(\theta + 180°) = \dfrac{\sqrt{3}}{3}$

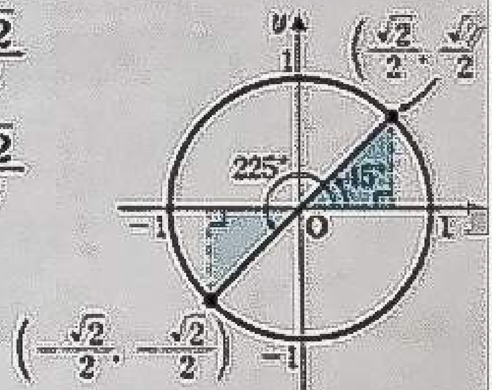


(1) $\theta = 45°$

[Sol] $\sin\theta = \dfrac{\sqrt{2}}{2}$ $\quad \sin(\theta + 180°) = -\dfrac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\sqrt{2}}{2}$ $\quad \cos(\theta + 180°) = -\dfrac{\sqrt{2}}{2}$

$\tan\theta = 1$ $\quad \tan(\theta + 180°) = 1$

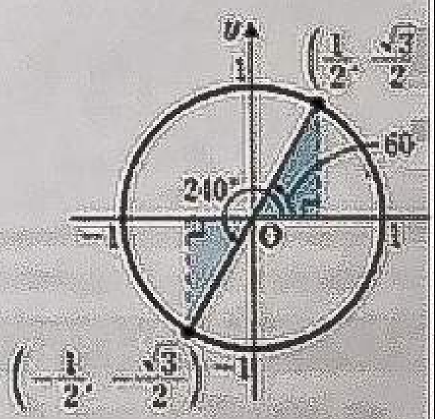


(2) $\theta = 60°$

[Sol] $\sin\theta = \dfrac{\sqrt{3}}{2}$ $\quad \sin(\theta + 180°) = -\dfrac{\sqrt{3}}{2}$

$\cos\theta = \dfrac{1}{2}$ $\quad \cos(\theta + 180°) = -\dfrac{1}{2}$

$\tan\theta = \sqrt{3}$ $\quad \tan(\theta + 180°) = \sqrt{3}$



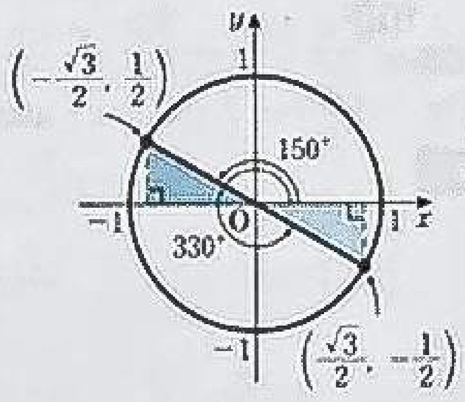
M105b

(3) $\theta=150^\circ$

[Sol] $\sin\theta=\dfrac{1}{2}$ $\qquad\sin(\theta+180^\circ)=-\dfrac{1}{2}$

$\cos\theta=-\dfrac{\sqrt{3}}{2}$ $\qquad\cos(\theta+180^\circ)=\dfrac{\sqrt{3}}{2}$

$\tan\theta=-\dfrac{\sqrt{3}}{3}$ $\qquad\tan(\theta+180^\circ)=-\dfrac{\sqrt{3}}{3}$

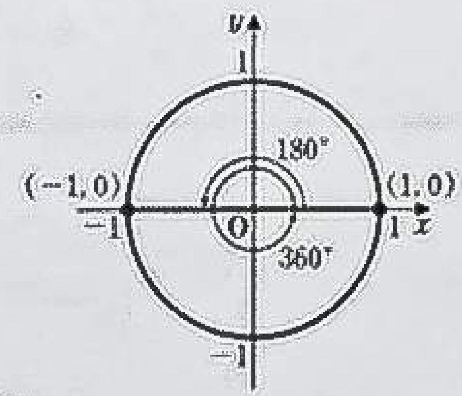


(4) $\theta=180^\circ$

[Sol] $\sin\theta=0$ $\qquad\sin(\theta+180^\circ)=0$

$\cos\theta=-1$ $\qquad\cos(\theta+180^\circ)=1$

$\tan\theta=0$ $\qquad\tan(\theta+180^\circ)=0$

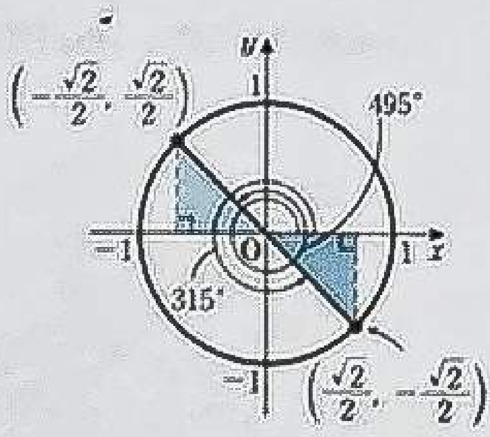


(5) $\theta=315^\circ$

[Sol] $\sin\theta=-\dfrac{\sqrt{2}}{2}$ $\qquad\sin(\theta+180^\circ)=\dfrac{\sqrt{2}}{2}$

$\cos\theta=\dfrac{\sqrt{2}}{2}$ $\qquad\cos(\theta+180^\circ)=-\dfrac{\sqrt{2}}{2}$

$\tan\theta=-1$ $\qquad\tan(\theta+180^\circ)=-1$



The relationships between trigonometric functions of $\theta+180^\circ$ and θ are as follows.

Trigonometric Functions of $\theta+180^\circ$

$\sin(\theta+180^\circ)=-\sin\theta$

$\cos(\theta+180^\circ)=-\cos\theta$

$\tan(\theta+180^\circ)=\tan\theta$

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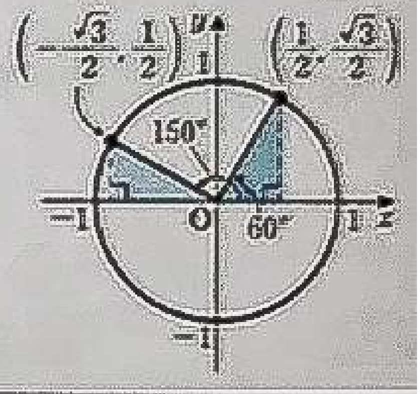
For each given θ, find the values of sin, cos and tan of θ and $\theta+90°$.

Ex. $\theta=60°$

[Sol] $\sin\theta=\frac{\sqrt{3}}{2}$ $\sin(\theta+90°)=\frac{1}{2}$

$\cos\theta=\frac{1}{2}$ $\cos(\theta+90°)=-\frac{\sqrt{3}}{2}$

$\tan\theta=\sqrt{3}$ $\tan(\theta+90°)=-\frac{\sqrt{3}}{3}$

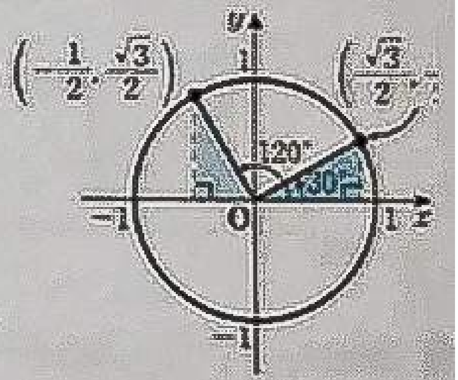


(1) $\theta=30°$

[Sol] $\sin\theta=\frac{1}{2}$ $\sin(\theta+90°)=\frac{\sqrt{3}}{2}$

$\cos\theta=\frac{\sqrt{3}}{2}$ $\cos(\theta+90°)=-\frac{1}{2}$

$\tan\theta=\frac{\sqrt{3}}{3}$ $\tan(\theta+90°)=-\sqrt{3}$

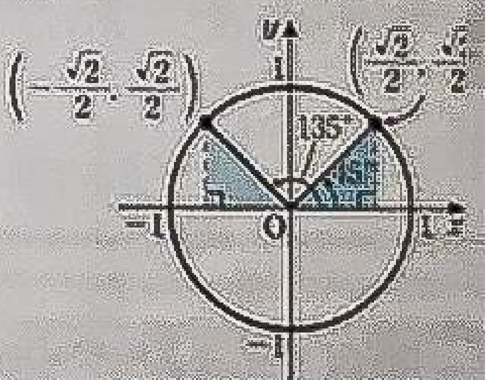


(2) $\theta=45°$

[Sol] $\sin\theta=\frac{\sqrt{2}}{2}$ $\sin(\theta+90°)=\frac{\sqrt{2}}{2}$

$\cos\theta=\frac{\sqrt{2}}{2}$ $\cos(\theta+90°)=-\frac{\sqrt{2}}{2}$

$\tan\theta=1$ $\tan(\theta+90°)=-1$



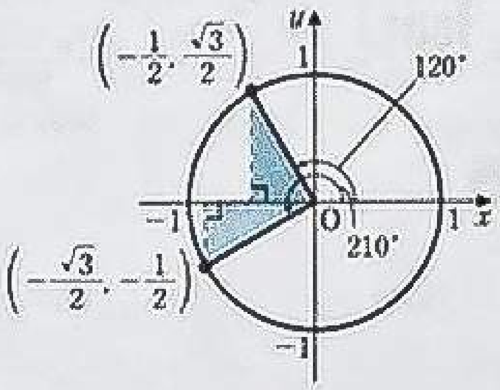
M106b

(3) $\theta=120^\circ$

[Sol] $\sin\theta=\frac{\sqrt{3}}{2}$ $\sin(\theta+90^\circ)=-\frac{1}{2}$

$\cos\theta=-\frac{1}{2}$ $\cos(\theta+90^\circ)=-\frac{\sqrt{3}}{2}$

$\tan\theta=-\sqrt{3}$ $\tan(\theta+90^\circ)=\frac{\sqrt{3}}{3}$

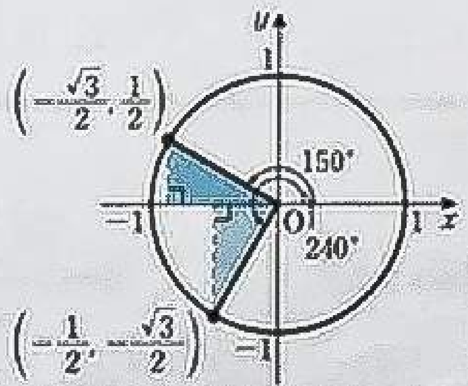


(4) $\theta=150^\circ$

[Sol] $\sin\theta=\frac{1}{2}$ $\sin(\theta+90^\circ)=-\frac{\sqrt{3}}{2}$

$\cos\theta=-\frac{\sqrt{3}}{2}$ $\cos(\theta+90^\circ)=-\frac{1}{2}$

$\tan\theta=-\frac{\sqrt{3}}{3}$ $\tan(\theta+90^\circ)=\sqrt{3}$

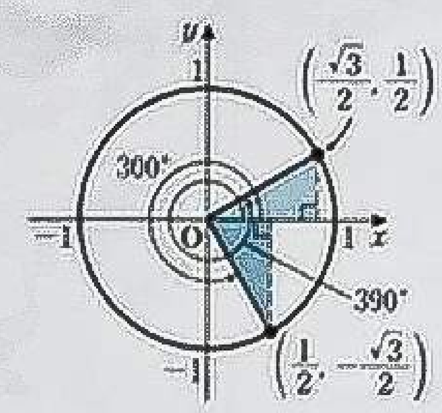


(5) $\theta=300^\circ$

[Sol] $\sin\theta=-\frac{\sqrt{3}}{2}$ $\sin(\theta+90^\circ)=\frac{1}{2}$

$\cos\theta=\frac{1}{2}$ $\cos(\theta+90^\circ)=\frac{\sqrt{3}}{2}$

$\tan\theta=-\sqrt{3}$ $\tan(\theta+90^\circ)=\frac{\sqrt{3}}{3}$



The relationships between trigonometric functions of $\theta+90^\circ$ and θ are as follows.

Trigonometric Functions of $\theta+90^\circ$

$\sin(\theta+90^\circ)=\cos\theta$

$\cos(\theta+90^\circ)=-\sin\theta$

$\tan(\theta+90^\circ)=-\frac{1}{\tan\theta}$

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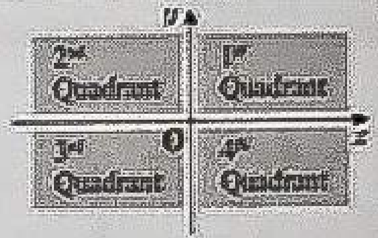
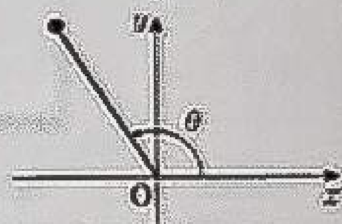
The four regions on a coordinate plane that are separated by the x-axis and y-axis are called the 1st *Quadrant*, 2nd *Quadrant*, 3rd *Quadrant* and 4th *Quadrant* as shown in the diagram on the right. (See Note)



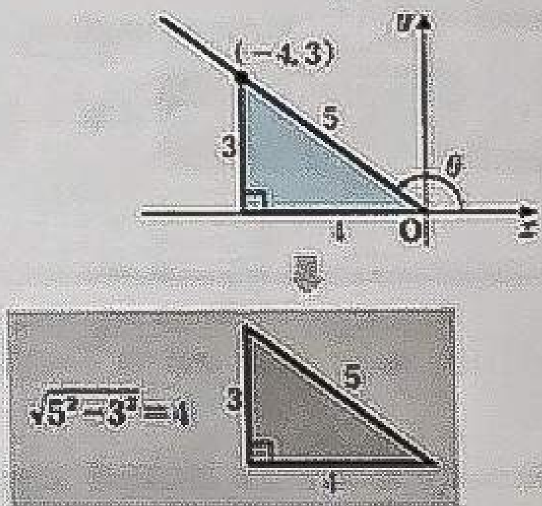
When the terminal side of θ is in the 2nd Quadrant, θ is called an ***angle in the 2nd Quadrant.***



Ex. Given that θ is an angle in the 2nd Quadrant and $\sin\theta = \frac{3}{5}$, find the values of $\cos\theta$ and $\tan\theta$.

[Sol] $\cos\theta = -\frac{4}{5}$

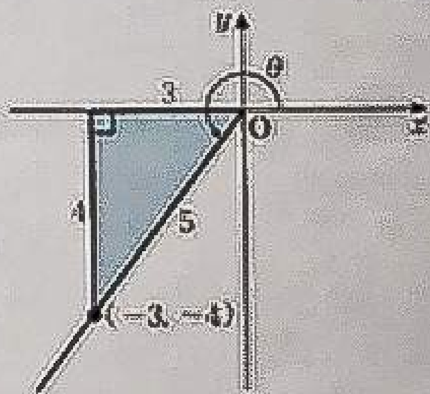
$\tan\theta = -\frac{3}{4}$



1. Given that θ is an angle in the 3rd Quadrant and $\sin\theta = -\frac{4}{5}$, find the values of $\cos\theta$ and $\tan\theta$.

[Sol] $\cos\theta = -\frac{3}{5}$

$\tan\theta = \frac{4}{3}$

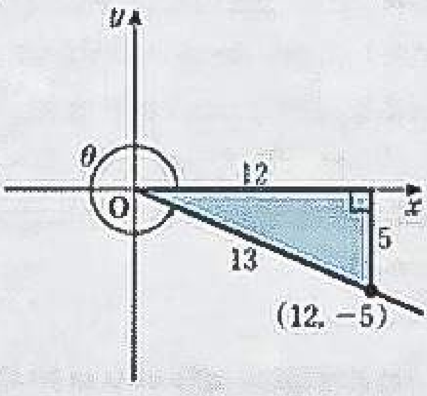


Note Points on the x-axis and y-axis do not belong to any quadrant.

M107b

2. Given that θ is an angle in the 4^{th} Quadrant and $\sin\theta = -\frac{5}{13}$, find the values of $\cos\theta$ and $\tan\theta$.

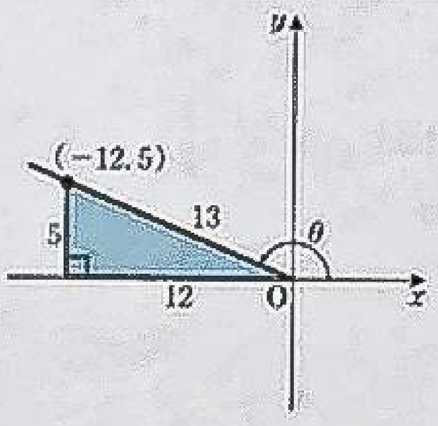
[Sol] $\cos\theta = \frac{12}{13}$



$\tan\theta = -\frac{5}{12}$

3. Given that θ is an angle in the 2^{nd} Quadrant and $\cos\theta = -\frac{12}{13}$, find the values of $\sin\theta$ and $\tan\theta$.

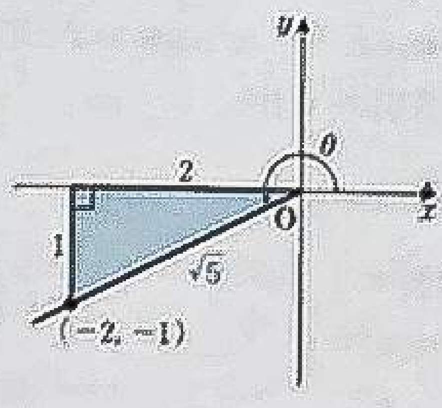
[Sol] $\sin\theta = \frac{5}{13}$



$\tan\theta = -\frac{5}{12}$

4. Given that θ is an angle in the 3^{rd} Quadrant and $\tan\theta = \frac{1}{2}$, find the values of $\sin\theta$ and $\cos\theta$.

[Sol] $\sin\theta = -\frac{\sqrt{5}}{5}$



$\cos\theta = -\frac{2\sqrt{5}}{5}$

Properties of Trigonometric Functions 1

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As with Trigonometric Ratios, the following formulas are true for Trigonome Functions.

Trigonometric Identities

$\sin^2\theta+\cos^2\theta=1$

$\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ $\quad 1+\tan^2\theta=\dfrac{1}{\cos^2\theta}$

Using the formulas above, solve the following questions.

Ex. Given that θ is an angle in the 3rd Quadrant and $\cos\theta=-\dfrac{4}{5}$, find the val of $\sin\theta$.

[Sol] Since θ is an angle in the 3rd Quadrant, $\sin\theta<0$

$$\therefore \sin\theta=-\sqrt{1-\left(-\frac{4}{5}\right)^2} \quad \leftarrow \boxed{\text{Since } \sin\theta<0,\ \sin\theta=-\sqrt{1-\cos^2\theta}}$$

$$=-\frac{3}{5}$$

(1) Given that θ is an angle in the 4th Quadrant and $\sin\theta=-\dfrac{1}{3}$, find the v of $\cos\theta$.

[Sol] Since θ is an angle in the 4th Quadrant, $\cos\theta>0$

$$\therefore \cos\theta=\sqrt{1-\left(-\frac{1}{3}\right)^2}$$

$$=\frac{2\sqrt{2}}{3}$$

(2) Given that θ is an angle in the 3rd Quadrant and $\cos\theta=-\dfrac{5}{13}$, find the v of $\sin\theta$.

[Sol] Since θ is an angle in the 3rd Quadrant, $\sin\theta<0$

$$\therefore \sin\theta=-\sqrt{1-\left(-\frac{5}{13}\right)^2}$$

$$=-\frac{12}{13}$$

M108b

(3) Given that θ is an angle in the 2nd Quadrant and $\sin\theta=\frac{3}{5}$, find the values of $\cos\theta$ and $\tan\theta$.

[Sol] Since θ is an angle in the 2nd Quadrant,
$\cos\theta<0$

$$\therefore\ \cos\theta=-\sqrt{1-\left(\frac{3}{5}\right)^2}$$
$$=-\frac{4}{5}$$

Also, $\tan\theta=\frac{3}{5}\div\left(-\frac{4}{5}\right)$ ⟵ $\boxed{\tan\theta=\frac{\sin\theta}{\cos\theta}}$
$$=-\frac{3}{4}$$

Alternative Solution

Since θ is an angle in the 2nd Quadrant,
$\cos\theta<0$

$$\therefore\ \cos\theta=-\sqrt{1-\left(\frac{3}{5}\right)^2}$$
$$=-\frac{4}{5}$$

Also, $\tan^2\theta=\frac{1}{\left(-\frac{4}{5}\right)^2}-1$ ⟵ $\boxed{1+\tan^2\theta=\frac{1}{\cos^2\theta}}$
$$=\frac{9}{16}$$

Since θ is an angle in the 2nd Quadrant,
$\tan\theta<0$

$$\therefore\ \tan\theta=-\frac{3}{4}$$

(4) Given that θ is an angle in the 3rd Quadrant and $\cos\theta=-\frac{1}{2}$, find the values of $\sin\theta$ and $\tan\theta$.

[Sol] Since θ is an angle in the 3rd Quadrant,
$\sin\theta<0$

$$\therefore\ \sin\theta=-\sqrt{1-\left(-\frac{1}{2}\right)^2}$$
$$=-\frac{\sqrt{3}}{2}$$

Also, $\tan\theta=-\frac{\sqrt{3}}{2}\div\left(-\frac{1}{2}\right)$
$$=\sqrt{3}$$

Alternative Solution

Since θ is an angle in the 3rd Quadrant,
$\sin\theta<0$

$$\therefore\ \sin\theta=-\sqrt{1-\left(-\frac{1}{2}\right)^2}$$
$$=-\frac{\sqrt{3}}{2}$$

Also, $\tan^2\theta=\frac{1}{\left(-\frac{1}{2}\right)^2}-1$
$$=3$$

Since θ is an angle in the 3rd Quadrant,
$\tan\theta>0$

$$\therefore\ \tan\theta=\sqrt{3}$$

(5) Given that θ is an angle in the 4th Quadrant and $\tan\theta=-2$, find the values of $\cos\theta$ and $\sin\theta$.

[Sol] $\frac{1}{\cos^2\theta}=1+(-2)^2$ ⟵ $\boxed{1+\tan^2\theta=\frac{1}{\cos^2\theta}}$
$$=5$$

$$\therefore\ \cos^2\theta=\frac{1}{5}$$

Since θ is an angle in the 4th Quadrant, $\cos\theta>0$

$$\therefore\ \cos\theta=\frac{\sqrt{5}}{5}$$

Also, $\sin\theta=-2\cdot\frac{\sqrt{5}}{5}$ ⟵ $\boxed{\text{Since } \tan\theta=\frac{\sin\theta}{\cos\theta},\ \sin\theta=\tan\theta\cos\theta}$
$$=-\frac{2\sqrt{5}}{5}$$

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Ex. Given $\sin\theta+\cos\theta=\frac{1}{2}$, find the values of the following expressions.

(1) $\sin\theta\cos\theta$

[Sol] Squaring both sides of $\sin\theta+\cos\theta=\frac{1}{2}$,

$$\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta=\frac{1}{4}$$

$\sin^2\theta+\cos^2\theta=1$

$$1+2\sin\theta\cos\theta=\frac{1}{4}$$

$$\therefore\ \sin\theta\cos\theta=-\frac{3}{8}$$

a^3+b^3
$=(a+b)(a^2-ab+b^2)$

(2) $\sin^3\theta+\cos^3\theta$

[Sol] $\sin^3\theta+\cos^3\theta=(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)$

$$=\frac{1}{2}\left[1-\left(-\frac{3}{8}\right)\right]$$

$\sin\theta+\cos\theta=\frac{1}{2}$

From (1), $\sin\theta\cos\theta=-\frac{3}{8}$

$$=\frac{11}{16}$$

1. Given $\sin\theta+\cos\theta=\frac{\sqrt{2}}{2}$, find the values of the following expressions.

(1) $\sin\theta\cos\theta$

[Sol] Squaring both sides of $\sin\theta+\cos\theta=\frac{\sqrt{2}}{2}$,

$$\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta=\frac{1}{2}$$

$$1+2\sin\theta\cos\theta=\frac{1}{2}$$

$$\therefore\ \sin\theta\cos\theta=-\frac{1}{4}$$

(2) $\sin^3\theta+\cos^3\theta$

[Sol] $\sin^3\theta+\cos^3\theta=(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)$

$$=\frac{\sqrt{2}}{2}\left[1-\left(-\frac{1}{4}\right)\right]$$

$$=\frac{5\sqrt{2}}{8}$$

M109b

2. Given $\sin\theta-\cos\theta=\frac{1}{5}$, find the values of the following expressions.

(1) $\sin\theta\cos\theta$

[Sol] Squaring both sides of $\sin\theta-\cos\theta=\frac{1}{5}$,

$$\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta=\frac{1}{25}$$

$$1-2\sin\theta\cos\theta=\frac{1}{25}$$

$$\therefore\ \sin\theta\cos\theta=\frac{12}{25}$$

(2) $\sin^3\theta-\cos^3\theta$

[Sol] $\sin^3\theta-\cos^3\theta=(\sin\theta-\cos\theta)(\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta)$

$$=\frac{1}{5}\left(1+\frac{12}{25}\right)$$

$$=\frac{37}{125}$$

a^3-b^3
$=(a-b)(a^2+ab+b^2)$

3. Given $\sin\theta+\cos\theta=-\frac{1}{3}$, find the value of $\sin^4\theta+\cos^4\theta$.

[Sol] Squaring both sides of $\sin\theta+\cos\theta=-\frac{1}{3}$,

$$\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta=\frac{1}{9}$$

$$1+2\sin\theta\cos\theta=\frac{1}{9}$$

$$\therefore\ \sin\theta\cos\theta=-\frac{4}{9}$$

$$\therefore\ \sin^4\theta+\cos^4\theta=(\sin^2\theta+\cos^2\theta)^2-2\sin^2\theta\cos^2\theta$$

$$=1^2-2\left(-\frac{4}{9}\right)^2$$

$$=\frac{49}{81}$$

a^4+b^4
$=(a^2+b^2)^2-2a^2b^2$

Properties of Trigonometric Functions 1

100%	~90%	~80%	~70%	69%~
0	—	1	—	2

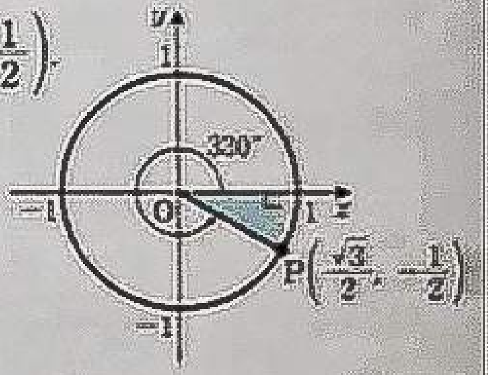
Name

Date / /

Time : to :

1. Place point P on the circumference of a circle with radius 1 such that gen
angle $\theta = 330°$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$. ➡ M

[Sol] The coordinates of point P are $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.



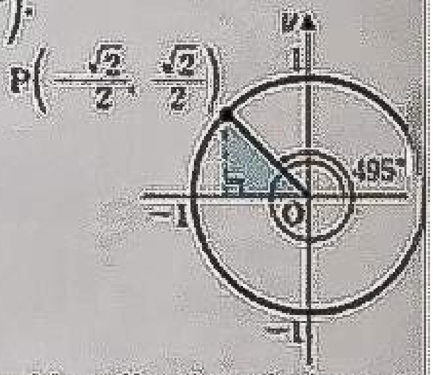
$\therefore\ \sin\theta = -\frac{1}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\tan\theta = -\frac{\sqrt{3}}{3}$

2. Place point P on the circumference of a circle with radius 1 such that gen
angle $\theta = 495°$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$. ➡ M I

[Sol] The coordinates of point P are $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.



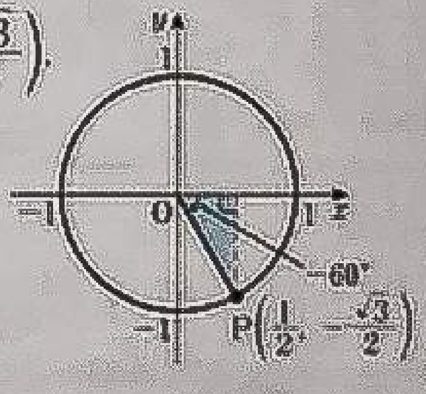
$\therefore\ \sin\theta = \frac{\sqrt{2}}{2}$

$\cos\theta = -\frac{\sqrt{2}}{2}$

$\tan\theta = -1$

3. Place point P on the circumference of a circle with radius 1 such that gen
angle $\theta = -60°$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$. ➡ M

[Sol] The coordinates of point P are $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.



$\therefore\ \sin\theta = -\frac{\sqrt{3}}{2}$

$\cos\theta = \frac{1}{2}$

$\tan\theta = -\sqrt{3}$

M110b

4. Given that θ is an angle in the 3rd Quadrant and $\sin\theta=-\frac{3}{5}$, find the values of $\cos\theta$ and $\tan\theta$.

➡ M108

[Sol] Since θ is an angle in the 3rd Quadrant, $\cos\theta<0$

$$\therefore \cos\theta=-\sqrt{1-\left(-\frac{3}{5}\right)^2}$$
$$=-\frac{4}{5}$$

$$\text{Also, } \tan\theta=\left(-\frac{3}{5}\right)\div\left(-\frac{4}{5}\right)$$
$$=\frac{3}{4}$$

Alternative Solution 1

Since θ is an angle in the 3rd Quadrant, $\cos\theta<0$

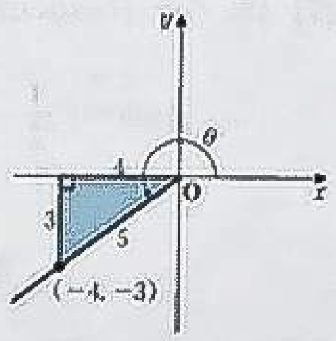
$$\therefore \cos\theta=-\sqrt{1-\left(-\frac{3}{5}\right)^2}=-\frac{4}{5}$$

$$\text{Also, } \tan^2\theta=\frac{1}{\left(-\frac{4}{5}\right)^2}-1=\frac{9}{16}$$

Since θ is an angle in the 3rd Quadrant, $\tan\theta>0$

$$\therefore \tan\theta=\frac{3}{4}$$

Alternative Solution 2



$$\cos\theta=-\frac{4}{5}$$
$$\tan\theta=\frac{3}{4}$$

5. Given that θ is an angle in the 4th Quadrant and $\cos\theta=\frac{5}{13}$, find the values of $\sin\theta$ and $\tan\theta$.

➡ M108

[Sol] Since θ is an angle in the 4th Quadrant, $\sin\theta<0$

$$\therefore \sin\theta=-\sqrt{1-\left(\frac{5}{13}\right)^2}$$
$$=-\frac{12}{13}$$

$$\text{Also, } \tan\theta=\left(-\frac{12}{13}\right)\div\frac{5}{13}$$
$$=-\frac{12}{5}$$

Alternative Solution 1

Since θ is an angle in the 4th Quadrant, $\sin\theta<0$

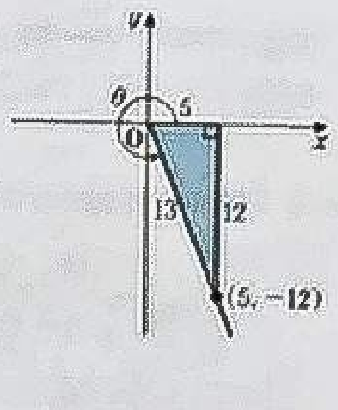
$$\therefore \sin\theta=-\sqrt{1-\left(\frac{5}{13}\right)^2}=-\frac{12}{13}$$

$$\text{Also, } \tan^2\theta=\frac{1}{\left(\frac{5}{13}\right)^2}-1=\frac{144}{25}$$

Since θ is an angle in the 4th Quadrant, $\tan\theta<0$

$$\therefore \tan\theta=-\frac{12}{5}$$

Alternative Solution 2



$$\sin\theta=-\frac{12}{13}$$
$$\tan\theta=-\frac{12}{5}$$

Properties of Trigonometric Functions 2

100%	~90%	~80%	~70%	69%~
(mistakes) 0	1~2	3~4	5~6	7~

Name

Date / /

Time : to :

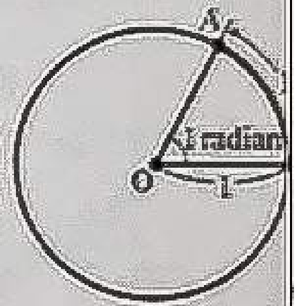
Given circle O with radius 1 in the diagram below, let A and B be the two poi[nts] on the circumference of circle O such that arc AB has length 1. Then, the size [of] the central angle is defined as 1 ***radian***.

The circumference of a circle with radius 1 is 2π.

Therefore, $360° = 2\pi$ radians,

i.e. $180° = \pi$ radians.

Circumference $= 2 \times$ radius $\times \pi$

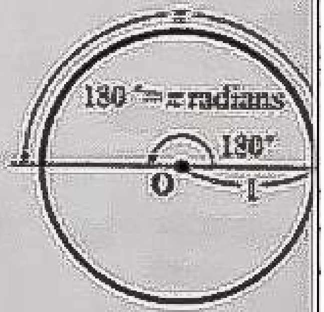


Measuring an angle in radians is called ***circular measure***.

Circular Measure

Since $180° = \pi$ radians,

$$1° = \frac{\pi}{180} \text{ radians,} \qquad 1 \text{ radian} = \frac{180°}{\pi}$$



Generally, the unit "radians" is omitted.

1. Convert the following angles from degrees to radians.

Ex. $30° = \frac{\pi}{6}$ ⟵ $30° = 30 \cdot 1° = 30 \cdot \frac{\pi}{180}$ (radians)

(1) $45° = \frac{\pi}{4}$

(2) $60° = \frac{\pi}{3}$

(3) $90° = \frac{\pi}{2}$

(4) $120° = \frac{2}{3}\pi$

(5) $135° = \frac{3}{4}\pi$

(6) $150° = \frac{5}{6}\pi$

(7) $210° = \frac{7}{6}\pi$

(8) $315° = \frac{7}{4}\pi$

(9) $-135° = -\frac{3}{4}\pi$

(10) $-240° = -\frac{4}{3}\pi$

Measuring an angle in degrees (°), for example 30° and 60°, is called ***degree measure***.

M111b

2. Convert the following angles from radians to degrees.

Ex $\frac{\pi}{3}=60^\circ$ ⟵ $\frac{\pi}{3}=\frac{\pi}{3}\cdot 1\,(\text{radians})=\frac{\pi}{3}\cdot\frac{180^\circ}{\pi}$

(1) $\frac{\pi}{6}=30^\circ$

(2) $\frac{\pi}{4}=45^\circ$

(3) $\frac{\pi}{2}=90^\circ$

(4) $\frac{2}{3}\pi=120^\circ$

(5) $\pi=180^\circ$

(6) $\frac{5}{4}\pi=225^\circ$

(7) $\frac{4}{3}\pi=240^\circ$

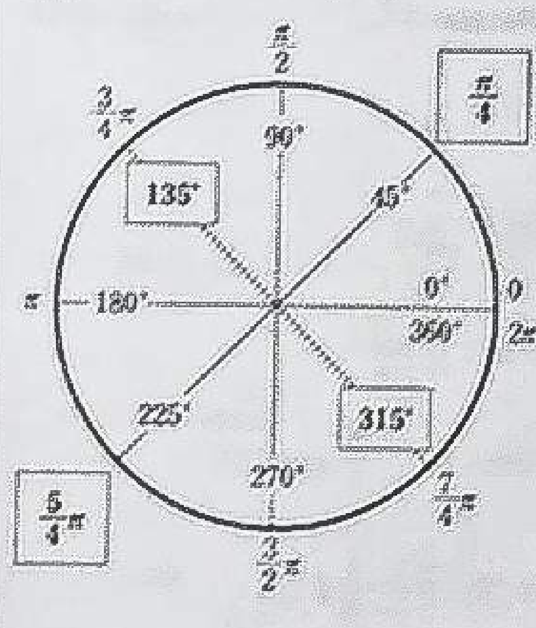
(8) $\frac{13}{6}\pi=390^\circ$

(9) $-\frac{\pi}{6}=-30^\circ$

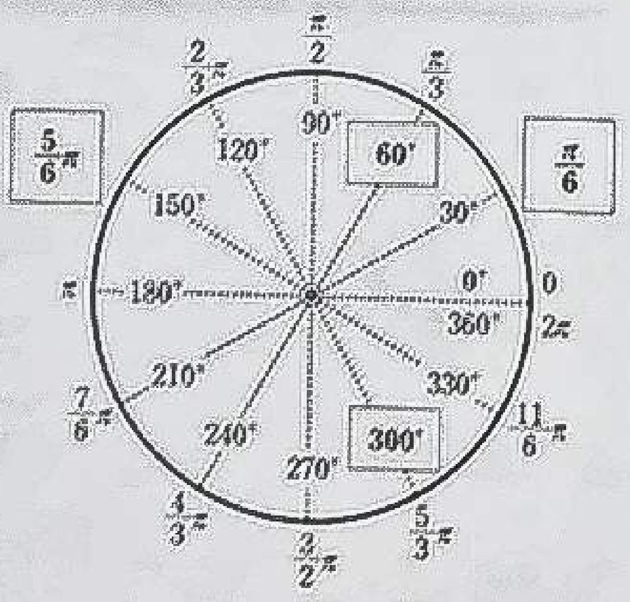
(10) $-\frac{11}{6}\pi=-330^\circ$

3. Fill in the following blanks.

(1)



(2)



M I I 2a

KUMON

Properties of Trigonometric Functions 2

100%	~90%	~80%	~70%	69%~

Name

Date / /

Time : to :

Ex. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=\frac{\pi}{6}$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

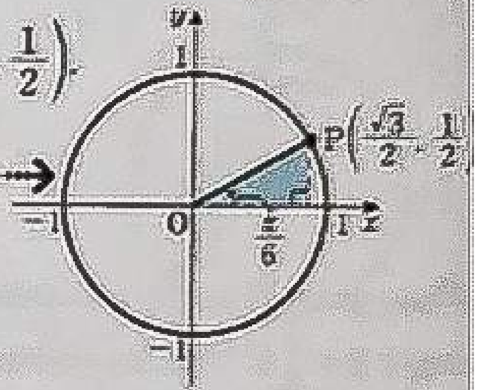
[Sol] The coordinates of point P are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

$\therefore\ \sin\theta=\frac{1}{2}$

$\cos\theta=\frac{\sqrt{3}}{2}$

$\tan\theta=\frac{\sqrt{3}}{3}$

$\frac{\pi}{6}=30^\circ$



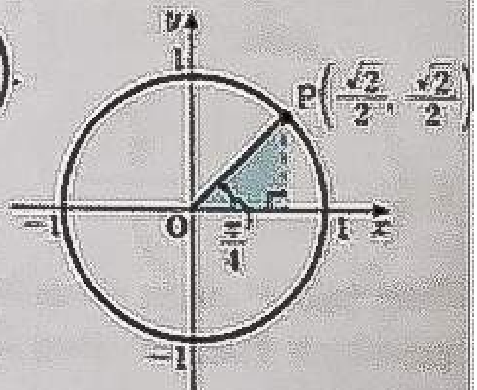
1. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=\frac{\pi}{4}$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

$\therefore\ \sin\theta=\frac{\sqrt{2}}{2}$

$\cos\theta=\frac{\sqrt{2}}{2}$

$\tan\theta=1$



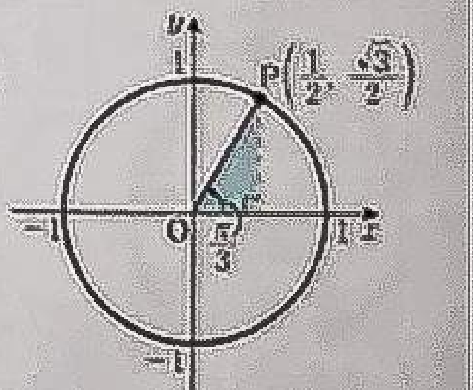
2. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=\frac{\pi}{3}$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$\therefore\ \sin\theta=\frac{\sqrt{3}}{2}$

$\cos\theta=\frac{1}{2}$

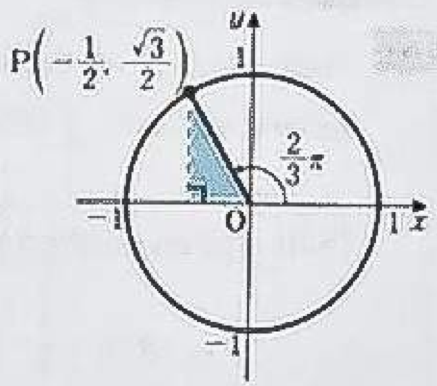
$\tan\theta=\sqrt{3}$



M I 12b

3. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=\frac{2}{3}\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



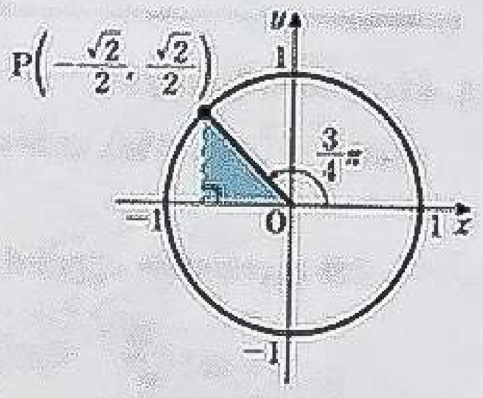
$\therefore\ \sin\theta=\frac{\sqrt{3}}{2}$

$\cos\theta=-\frac{1}{2}$

$\tan\theta=-\sqrt{3}$

4. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=\frac{3}{4}\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.



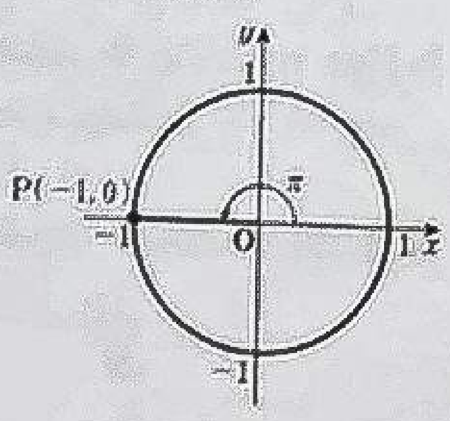
$\therefore\ \sin\theta=\frac{\sqrt{2}}{2}$

$\cos\theta=-\frac{\sqrt{2}}{2}$

$\tan\theta=-1$

5. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $(-1, 0)$.



$\therefore\ \sin\theta=0$

$\cos\theta=-1$

$\tan\theta=0$

M I I 3a

KUMON

M I I 3

Properties of Trigonometric Functions 2

Name

Date / /

Time : to :

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(mistakes) 0	—	1	—	2

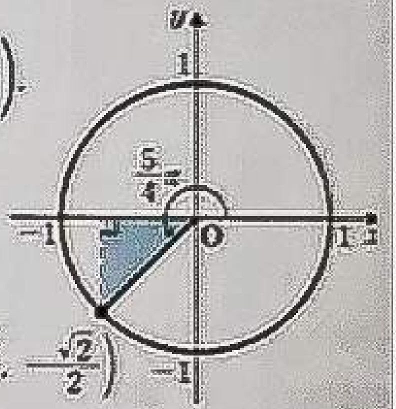
1. Place point P on the circumference of a circle with radius 1 such that gen
 angle $\theta=\frac{5}{4}\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

$\therefore\ \sin\theta=-\frac{\sqrt{2}}{2}$

$\cos\theta=-\frac{\sqrt{2}}{2}$

$\tan\theta=1$



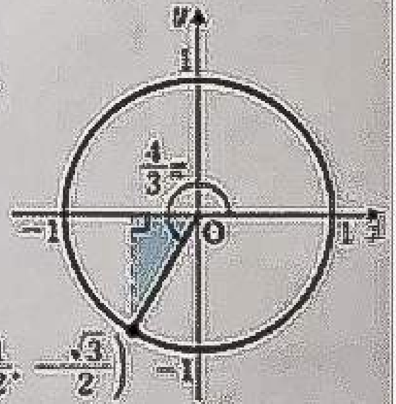
2. Place point P on the circumference of a circle with radius 1 such that gene
 angle $\theta=\frac{4}{3}\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

$\therefore\ \sin\theta=-\frac{\sqrt{3}}{2}$

$\cos\theta=-\frac{1}{2}$

$\tan\theta=\sqrt{3}$



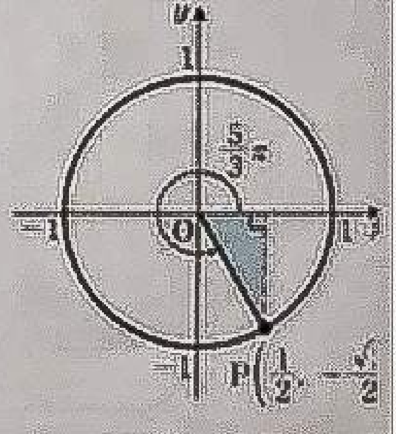
3. Place point P on the circumference of a circle with radius 1 such that gen
 angle $\theta=\frac{5}{3}\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

$\therefore\ \sin\theta=-\frac{\sqrt{3}}{2}$

$\cos\theta=\frac{1}{2}$

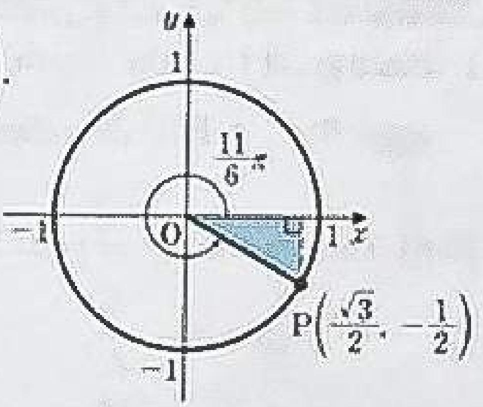
$\tan\theta=-\sqrt{3}$



M113b

4. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=\frac{11}{6}\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.



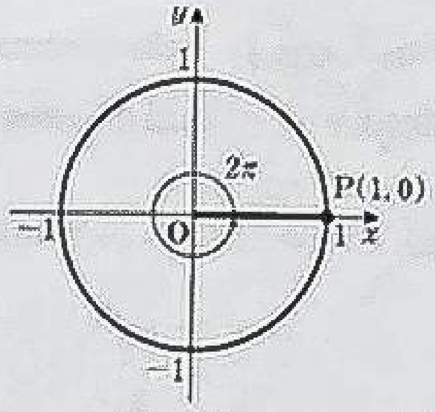
$\therefore\ \sin\theta=-\frac{1}{2}$

$\cos\theta=\frac{\sqrt{3}}{2}$

$\tan\theta=-\frac{\sqrt{3}}{3}$

5. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=2\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $(1, 0)$.



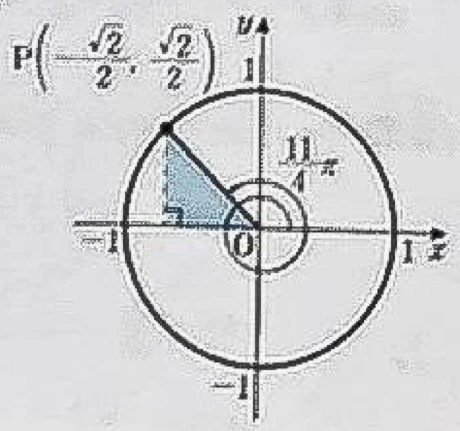
$\therefore\ \sin\theta=0$

$\cos\theta=1$

$\tan\theta=0$

6. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=\frac{11}{4}\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.



$\therefore\ \sin\theta=\frac{\sqrt{2}}{2}$

$\cos\theta=-\frac{\sqrt{2}}{2}$

$\tan\theta=-1$

M I 14a

KUMON

Properties of Trigonometric Functions 2

Name

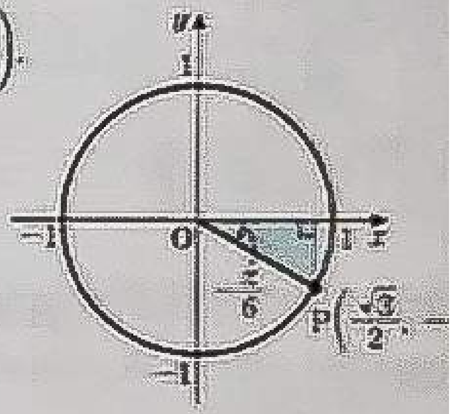
Date / /

Time : to :

100%	-90%	-80%	-70%	69%-
mistakes 0	—	1	—	2

1. Place point P on the circumference of a circle with radius 1 such that gene
angle $\theta=-\frac{\pi}{6}$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.



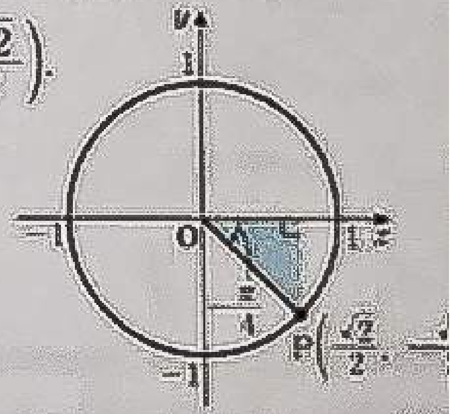
$\therefore\ \sin\theta=-\frac{1}{2}$

$\cos\theta=\frac{\sqrt{3}}{2}$

$\tan\theta=-\frac{\sqrt{3}}{3}$

2. Place point P on the circumference of a circle with radius 1 such that gene
angle $\theta=-\frac{\pi}{4}$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.



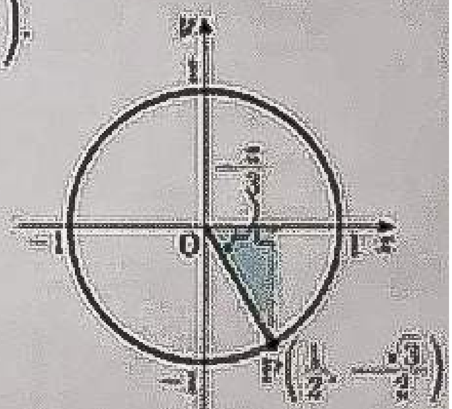
$\therefore\ \sin\theta=-\frac{\sqrt{2}}{2}$

$\cos\theta=\frac{\sqrt{2}}{2}$

$\tan\theta=-1$

3. Place point P on the circumference of a circle with radius 1 such that gene
angle $\theta=-\frac{\pi}{3}$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.



$\therefore\ \sin\theta=-\frac{\sqrt{3}}{2}$

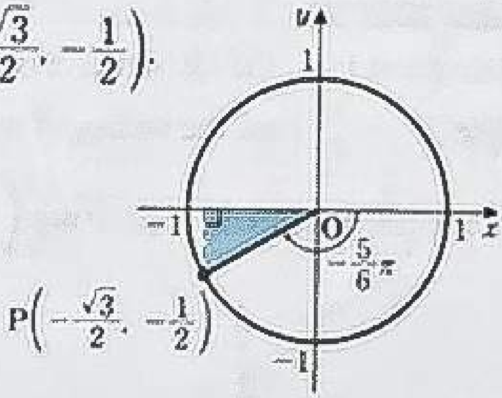
$\cos\theta=\frac{1}{2}$

$\tan\theta=-\sqrt{3}$

M114b

4. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=-\frac{5}{6}\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.



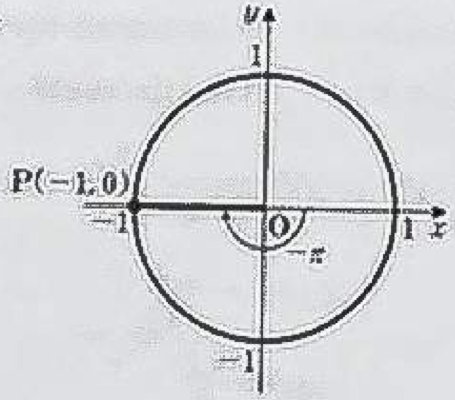
$\therefore \sin\theta=-\frac{1}{2}$

$\cos\theta=-\frac{\sqrt{3}}{2}$

$\tan\theta=\frac{\sqrt{3}}{3}$

5. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=-\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $(-1, 0)$.



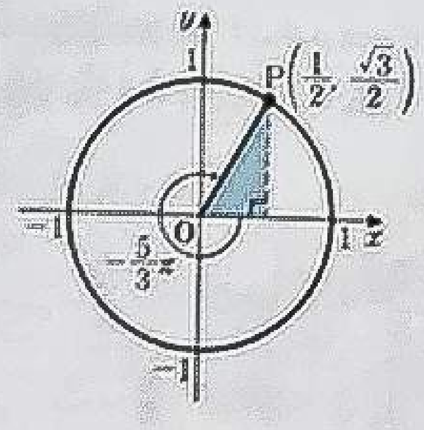
$\therefore \sin\theta=0$

$\cos\theta=-1$

$\tan\theta=0$

6. Place point P on the circumference of a circle with radius 1 such that general angle $\theta=-\frac{5}{3}\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] The coordinates of point P are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



$\therefore \sin\theta=\frac{\sqrt{3}}{2}$

$\cos\theta=\frac{1}{2}$

$\tan\theta=\sqrt{3}$

M I 15a KUMON

Properties of Trigonometric Functions 2

Name

Date / /

Time : to :

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(mistakes) 0	–	1	2	3–

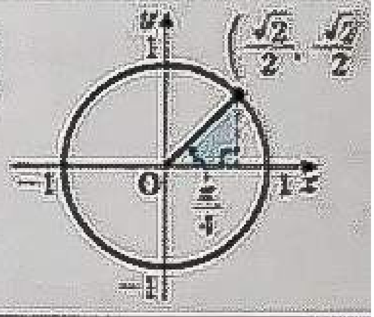
1. For each given angle θ, find the value of $\sin\left(\theta+\frac{\pi}{6}\right)$.

Ex. $\theta=\frac{\pi}{12}$

$\sin\left(\frac{\pi}{12}+\frac{\pi}{6}\right)=\sin\frac{\pi}{4}$

[Sol] Since $\frac{\pi}{12}+\frac{\pi}{6}=\frac{\pi}{4}$,

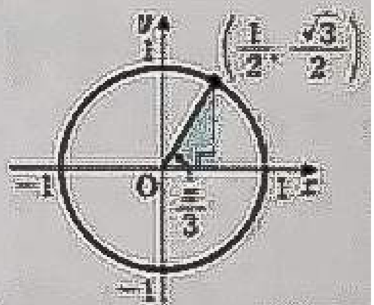
$\sin\left(\theta+\frac{\pi}{6}\right)=\frac{\sqrt{2}}{2}$



(1) $\theta=\frac{\pi}{6}$

[Sol] Since $\frac{\pi}{6}+\frac{\pi}{6}=\frac{\pi}{3}$,

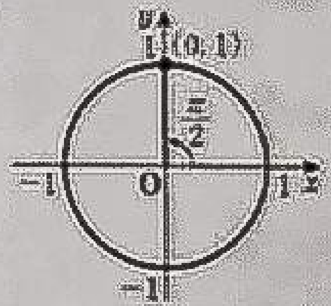
$\sin\left(\theta+\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$



(2) $\theta=\frac{\pi}{3}$

[Sol] Since $\frac{\pi}{3}+\frac{\pi}{6}=\frac{\pi}{2}$,

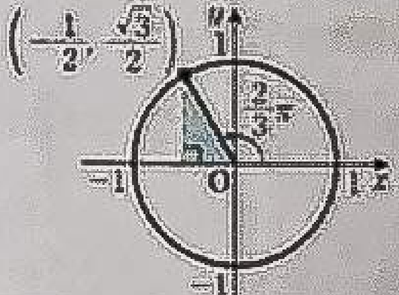
$\sin\left(\theta+\frac{\pi}{6}\right)=1$



(3) $\theta=\frac{\pi}{2}$

[Sol] Since $\frac{\pi}{2}+\frac{\pi}{6}=\frac{2}{3}\pi$,

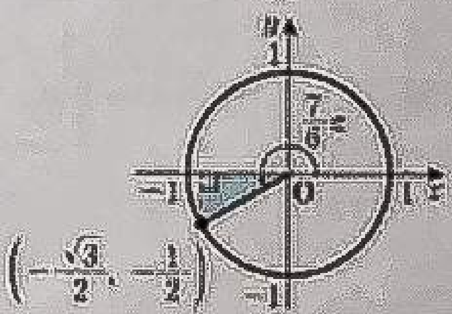
$\sin\left(\theta+\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$



(4) $\theta=\pi$

[Sol] Since $\pi+\frac{\pi}{6}=\frac{7}{6}\pi$,

$\sin\left(\theta+\frac{\pi}{6}\right)=-\frac{1}{2}$



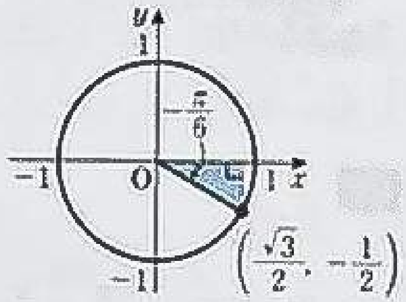
M115b

2. For each given angle θ, find the value of $\sin\left(\theta-\frac{\pi}{2}\right)$.

(1) $\theta=\frac{\pi}{3}$

[Sol] Since $\frac{\pi}{3}-\frac{\pi}{2}=-\frac{\pi}{6}$,

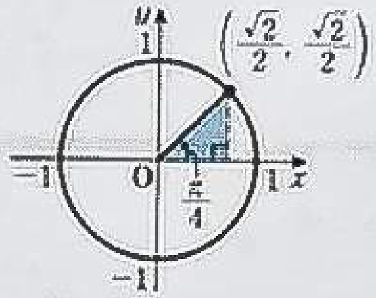
$$\sin\left(\theta-\frac{\pi}{2}\right)=-\frac{1}{2}$$



(2) $\theta=\frac{3}{4}\pi$

[Sol] Since $\frac{3}{4}\pi-\frac{\pi}{2}=\frac{\pi}{4}$,

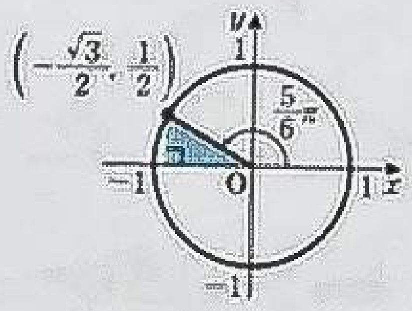
$$\sin\left(\theta-\frac{\pi}{2}\right)=\frac{\sqrt{2}}{2}$$



(3) $\theta=\frac{4}{3}\pi$

[Sol] Since $\frac{4}{3}\pi-\frac{\pi}{2}=\frac{5}{6}\pi$,

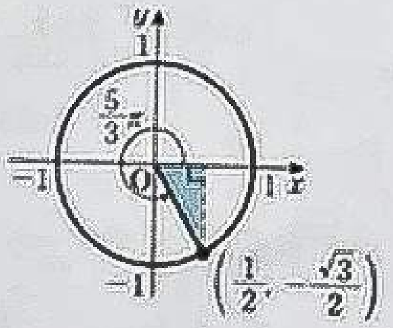
$$\sin\left(\theta-\frac{\pi}{2}\right)=\frac{1}{2}$$



(4) $\theta=\frac{13}{6}\pi$

[Sol] Since $\frac{13}{6}\pi-\frac{\pi}{2}=\frac{5}{3}\pi$,

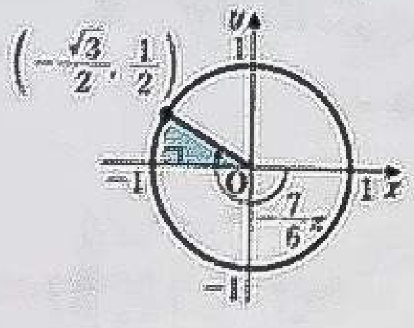
$$\sin\left(\theta-\frac{\pi}{2}\right)=-\frac{\sqrt{3}}{2}$$



(5) $\theta=-\frac{2}{3}\pi$

[Sol] Since $-\frac{2}{3}\pi-\frac{\pi}{2}=-\frac{7}{6}\pi$,

$$\sin\left(\theta-\frac{\pi}{2}\right)=\frac{1}{2}$$



Properties of Trigonometric Functions 2

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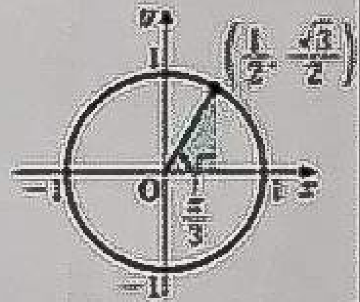
1. For each given angle θ, find the value of $\cos\left(\theta+\frac{\pi}{4}\right)$.

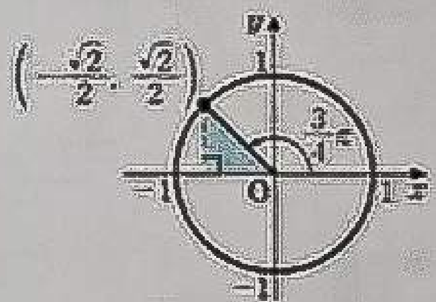
(1) $\theta=\frac{\pi}{12}$



[Sol] Since $\frac{\pi}{12}+\frac{\pi}{4}=\frac{\pi}{3}$,

$$\cos\left(\theta+\frac{\pi}{4}\right)=\frac{1}{2}$$

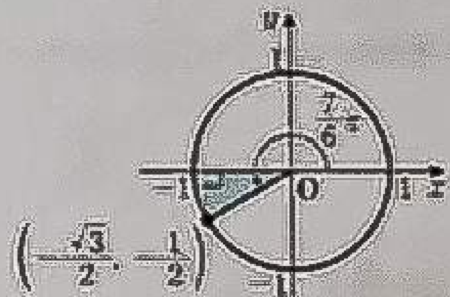
(2) $\theta=\frac{\pi}{2}$



[Sol] Since $\frac{\pi}{2}+\frac{\pi}{4}=\frac{3}{4}\pi$,

$$\cos\left(\theta+\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

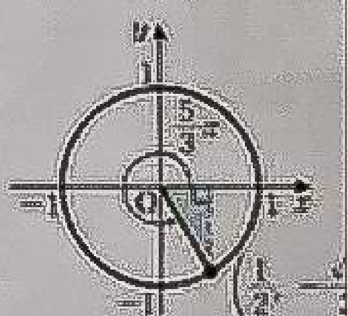
(3) $\theta=\frac{11}{12}\pi$



[Sol] Since $\frac{11}{12}\pi+\frac{\pi}{4}=\frac{7}{6}\pi$,

$$\cos\left(\theta+\frac{\pi}{4}\right)=-\frac{\sqrt{3}}{2}$$

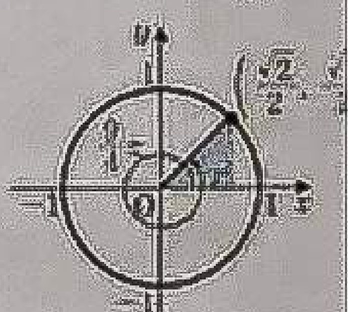
(4) $\theta=\frac{17}{12}\pi$



[Sol] Since $\frac{17}{12}\pi+\frac{\pi}{4}=\frac{5}{3}\pi$,

$$\cos\left(\theta+\frac{\pi}{4}\right)=\frac{1}{2}$$

(5) $\theta=2\pi$



[Sol] Since $2\pi+\frac{\pi}{4}=\frac{9}{4}\pi$,

$$\cos\left(\theta+\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

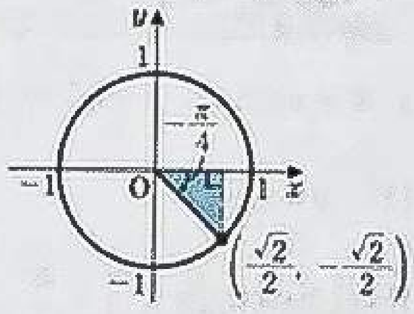
M116b

2. For each given angle θ, find the value of $\cos\left(2\theta-\frac{\pi}{2}\right)$.

(1) $\theta=\frac{\pi}{8}$

[Sol] Since $2\cdot\frac{\pi}{8}-\frac{\pi}{2}=-\frac{\pi}{4}$,

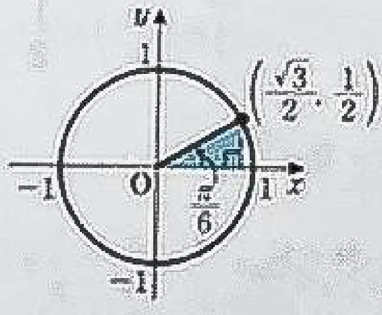
$$\cos\left(2\theta-\frac{\pi}{2}\right)=\frac{\sqrt{2}}{2}$$



(2) $\theta=\frac{\pi}{3}$

[Sol] Since $2\cdot\frac{\pi}{3}-\frac{\pi}{2}=\frac{\pi}{6}$,

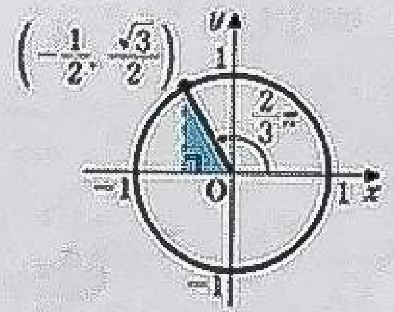
$$\cos\left(2\theta-\frac{\pi}{2}\right)=\frac{\sqrt{3}}{2}$$



(3) $\theta=\frac{7}{12}\pi$

[Sol] Since $2\cdot\frac{7}{12}\pi-\frac{\pi}{2}=\frac{2}{3}\pi$,

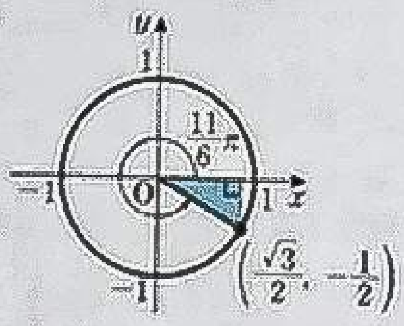
$$\cos\left(2\theta-\frac{\pi}{2}\right)=-\frac{1}{2}$$



(4) $\theta=\frac{7}{6}\pi$

[Sol] Since $2\cdot\frac{7}{6}\pi-\frac{\pi}{2}=\frac{11}{6}\pi$,

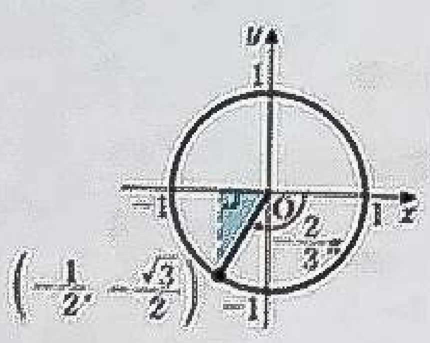
$$\cos\left(2\theta-\frac{\pi}{2}\right)=\frac{\sqrt{3}}{2}$$



(5) $\theta=-\frac{\pi}{12}$

[Sol] Since $2\cdot\left(-\frac{\pi}{12}\right)-\frac{\pi}{2}=-\frac{2}{3}\pi$,

$$\cos\left(2\theta-\frac{\pi}{2}\right)=-\frac{1}{2}$$



Properties of Trigonometric Functions 2

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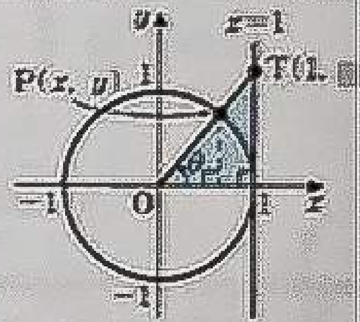
Let the point of intersection of the terminal side of angle θ and the unit circle be $\mathrm{P}(x, y)$. Also, let the point of intersection of line OP and line $x=1$ be $\mathrm{T}(1, m)$. Then, $\tan\theta = \frac{y}{x} = \frac{m}{1} = m$ is true.

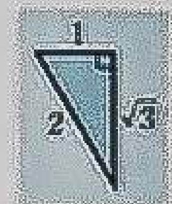


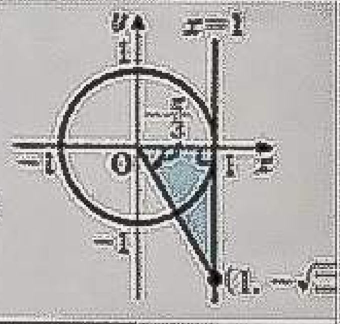
1. For each given angle θ, find the value of $\tan\left(\theta-\frac{\pi}{2}\right)$.

Ex. $\theta=\frac{\pi}{6}$

[Sol] Since $\frac{\pi}{6}-\frac{\pi}{2}=-\frac{\pi}{3}$,

$\tan\left(\theta-\frac{\pi}{2}\right)=-\sqrt{3}$

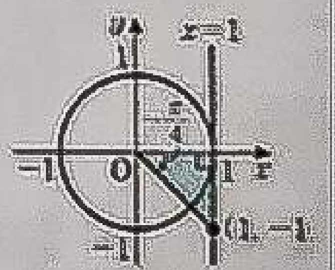




(1) $\theta=\frac{\pi}{4}$

[Sol] Since $\frac{\pi}{4}-\frac{\pi}{2}=-\frac{\pi}{4}$,

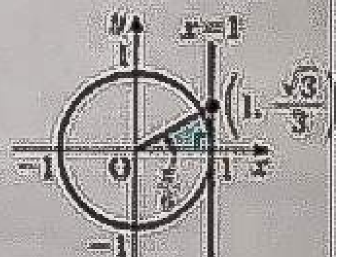
$\tan\left(\theta-\frac{\pi}{2}\right)=-1$



(2) $\theta=\frac{2}{3}\pi$

[Sol] Since $\frac{2}{3}\pi-\frac{\pi}{2}=\frac{\pi}{6}$,

$\tan\left(\theta-\frac{\pi}{2}\right)=\frac{\sqrt{3}}{3}$



M117b

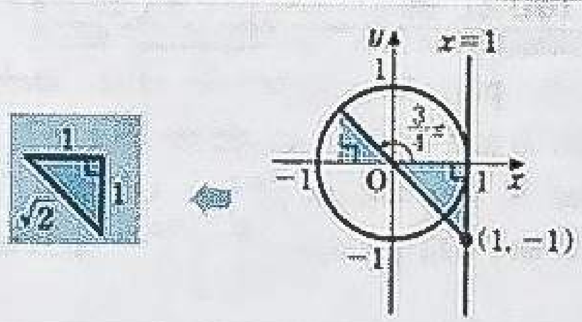
2. For each given angle θ, find the value of $\tan\left(\theta+\frac{\pi}{4}\right)$.

Ex

$\theta=\frac{\pi}{2}$

[Sol] Since $\frac{\pi}{2}+\frac{\pi}{4}=\frac{3}{4}\pi$,

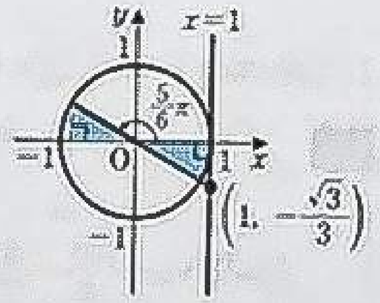
$$\tan\left(\theta+\frac{\pi}{4}\right)=-1$$



(1) $\theta=\frac{7}{12}\pi$

[Sol] Since $\frac{7}{12}\pi+\frac{\pi}{4}=\frac{5}{6}\pi$,

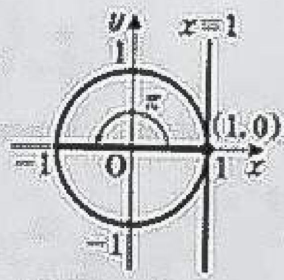
$$\tan\left(\theta+\frac{\pi}{4}\right)=-\frac{\sqrt{3}}{3}$$



(2) $\theta=\frac{3}{4}\pi$

[Sol] Since $\frac{3}{4}\pi+\frac{\pi}{4}=\pi$,

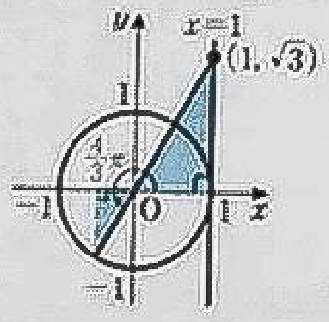
$$\tan\left(\theta+\frac{\pi}{4}\right)=0$$



(3) $\theta=\frac{13}{12}\pi$

[Sol] Since $\frac{13}{12}\pi+\frac{\pi}{4}=\frac{4}{3}\pi$,

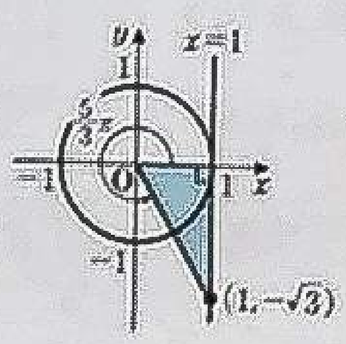
$$\tan\left(\theta+\frac{\pi}{4}\right)=\sqrt{3}$$



(4) $\theta=\frac{17}{12}\pi$

[Sol] Since $\frac{17}{12}\pi+\frac{\pi}{4}=\frac{5}{3}\pi$,

$$\tan\left(\theta+\frac{\pi}{4}\right)=-\sqrt{3}$$



M118a

KUMON

Properties of Trigonometric Functions 2

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Time : to :

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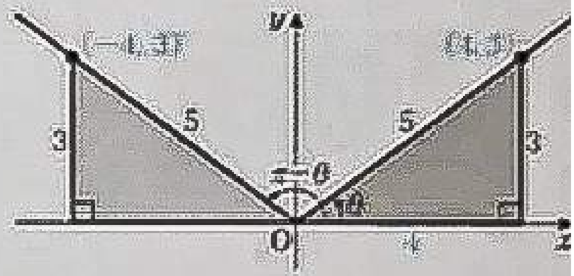
1. Given that θ is an angle in the 1st Quadrant and $\sin\theta=\frac{3}{5}$, evaluate the following expressions using the diagram below.



(1) $\cos\theta=\frac{4}{5}$

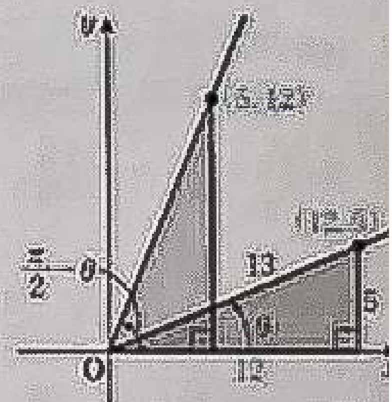
(2) $\tan\theta=\frac{3}{4}$

(3) $\sin(\pi-\theta)=\frac{3}{5}$

(4) $\cos(\pi-\theta)=-\frac{4}{5}$

(5) $\tan(\pi-\theta)=-\frac{3}{4}$

2. Given that θ is an angle in the 1st Quadrant and $\sin\theta=\frac{5}{13}$, evaluate the following expressions using the diagram below.



(1) $\cos\theta=\frac{12}{13}$

(2) $\tan\theta=\frac{5}{12}$

(3) $\sin\left(\frac{\pi}{2}-\theta\right)=\frac{12}{13}$

(4) $\cos\left(\frac{\pi}{2}-\theta\right)=\frac{5}{13}$

(5) $\tan\left(\frac{\pi}{2}-\theta\right)=\frac{12}{5}$

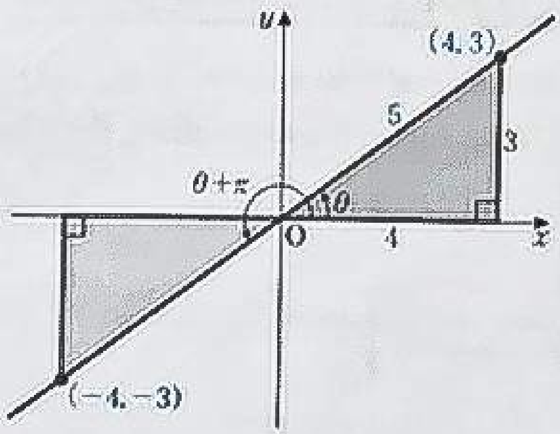
Using circular measure, the formulas in M98 can be expressed as follows:

$\sin(\pi-\theta)=\sin\theta,\quad \cos(\pi-\theta)=-\cos\theta,\quad \tan(\pi-\theta)=-\tan\theta$

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta,\quad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta,\quad \tan\left(\frac{\pi}{2}-\theta\right)=\frac{1}{\tan\theta}$

M118b

3. Given that θ is an angle in the 1st Quadrant and $\tan\theta=\frac{3}{4}$, evaluate the following expressions using the diagram below.



(1) $\sin\theta=\frac{3}{5}$

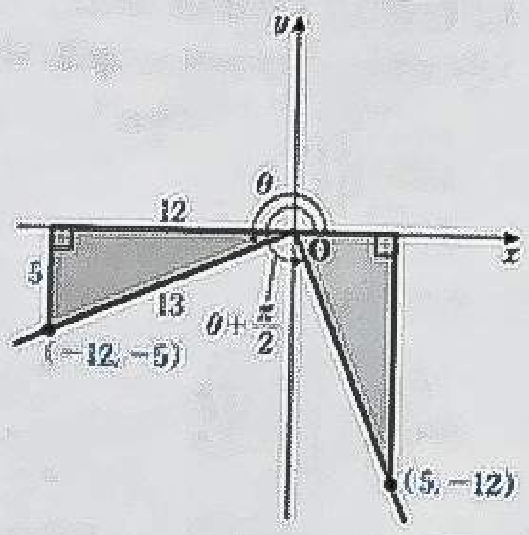
(2) $\cos\theta=\frac{4}{5}$

(3) $\sin(\theta+\pi)=-\frac{3}{5}$

(4) $\cos(\theta+\pi)=-\frac{4}{5}$

(5) $\tan(\theta+\pi)=\frac{3}{4}$

4. Given that θ is an angle in the 3rd Quadrant and $\cos\theta=-\frac{12}{13}$, evaluate the following expressions using the diagram below.



(1) $\sin\theta=-\frac{5}{13}$

(2) $\tan\theta=\frac{5}{12}$

(3) $\sin\left(\theta+\frac{\pi}{2}\right)=-\frac{12}{13}$

(4) $\cos\left(\theta+\frac{\pi}{2}\right)=\frac{5}{13}$

(5) $\tan\left(\theta+\frac{\pi}{2}\right)=-\frac{12}{5}$

Using circular measure, the formulas in M105 and M106 can be expressed as follows:

$\sin(\theta+\pi)=-\sin\theta,\quad \cos(\theta+\pi)=-\cos\theta,\quad \tan(\theta+\pi)=\tan\theta$

$\sin\left(\theta+\frac{\pi}{2}\right)=\cos\theta,\quad \cos\left(\theta+\frac{\pi}{2}\right)=-\sin\theta,\quad \tan\left(\theta+\frac{\pi}{2}\right)=-\frac{1}{\tan\theta}$

KUMON

Properties of Trigonometric Functions 2

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Prove the following identities.

Ex.

$$\frac{\sin\left(\theta+\frac{\pi}{2}\right)}{1-\sin\theta}+\frac{1+\sin(\theta+\pi)}{\cos\theta}=\frac{2}{\cos\theta}$$

[Sol] $\text{LHS}=\frac{\cos\theta}{1-\sin\theta}+\frac{1-\sin\theta}{\cos\theta}$ ← $\sin\left(\theta+\frac{\pi}{2}\right)=\cos\theta$, $\sin(\theta+\pi)=-\sin\theta$

$$=\frac{\cos^2\theta+(1-\sin\theta)^2}{(1-\sin\theta)\cos\theta}$$

$$=\frac{\cos^2\theta+1-2\sin\theta+\sin^2\theta}{(1-\sin\theta)\cos\theta}$$

$\sin^2\theta+\cos^2\theta=1$

$$=\frac{2(1-\sin\theta)}{(1-\sin\theta)\cos\theta}$$

$$=\frac{2}{\cos\theta}=\text{RHS}$$

(1) $\frac{\cos(-\theta)}{1-\sin\theta}-\frac{\cos(\theta+\pi)}{1+\sin\theta}=\frac{2}{\cos\theta}$

[Sol] $\text{LHS}=\frac{\cos\theta}{1-\sin\theta}+\frac{\cos\theta}{1+\sin\theta}$ ← $\cos(-\theta)=\cos\theta$, $\cos(\theta+\pi)=-\cos\theta$

$$=\frac{\cos\theta(1+\sin\theta)+\cos\theta(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$=\frac{2\cos\theta}{1-\sin^2\theta}$$

$$=\frac{2\cos\theta}{\cos^2\theta}$$

$$=\frac{2}{\cos\theta}=\text{RHS}$$

M119b

(2) $\tan(\theta+\pi)-\tan\left(\theta+\frac{\pi}{2}\right)=\frac{1}{\cos\theta\sin\theta}$

[Sol] $\text{LHS}=\tan\theta+\frac{1}{\tan\theta}$ ⇐ $\tan(\theta+\pi)=\tan\theta,\ \tan\left(\theta+\frac{\pi}{2}\right)=-\frac{1}{\tan\theta}$

$=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}$ ⇐ $\tan\theta=\frac{\sin\theta}{\cos\theta}$

$=\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}$

$=\frac{1}{\cos\theta\sin\theta}=\text{RHS}$

(3) $\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{1+\sin(\pi-\theta)}+\tan(\theta+\pi)=\frac{1}{\cos\theta}$

[Sol] $\text{LHS}=\frac{\cos\theta}{1+\sin\theta}+\tan\theta$ ⇐ $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta,\ \sin(\pi-\theta)=\sin\theta,$ $\tan(\theta+\pi)=\tan\theta$

$=\frac{\cos\theta}{1+\sin\theta}+\frac{\sin\theta}{\cos\theta}$

$=\frac{\cos^2\theta+\sin\theta(1+\sin\theta)}{(1+\sin\theta)\cos\theta}$

$=\frac{\cos^2\theta+\sin\theta+\sin^2\theta}{(1+\sin\theta)\cos\theta}$

$=\frac{1+\sin\theta}{(1+\sin\theta)\cos\theta}$

$=\frac{1}{\cos\theta}=\text{RHS}$

Properties of Trigonometric Functions 2

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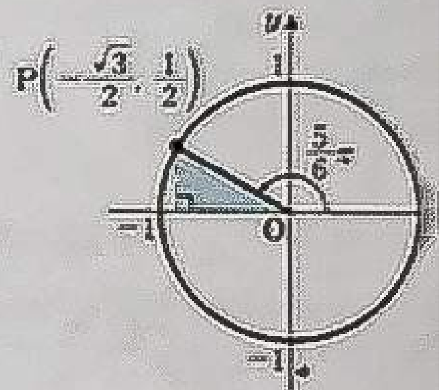
1. Place point P on the circumference of a circle with radius 1 such that gen
angle $\theta=\frac{5}{6}\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$. ➡ MI

[Sol] The coordinates of point P are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.



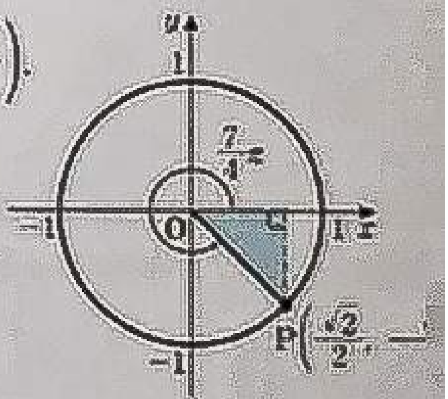
$\therefore\ \sin\theta=\frac{1}{2}$

$\cos\theta=-\frac{\sqrt{3}}{2}$

$\tan\theta=-\frac{\sqrt{3}}{3}$

2. Place point P on the circumference of a circle with radius 1 such that gen
angle $\theta=\frac{7}{4}\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$. ➡ MI

[Sol] The coordinates of point P are $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.



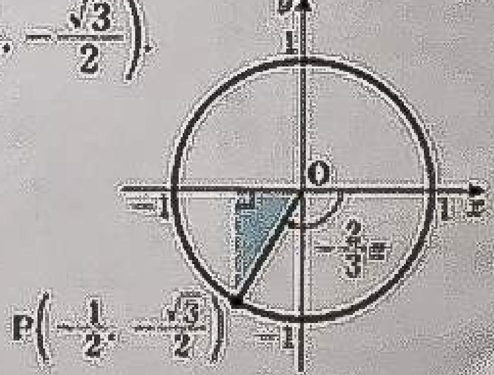
$\therefore\ \sin\theta=-\frac{\sqrt{2}}{2}$

$\cos\theta=\frac{\sqrt{2}}{2}$

$\tan\theta=-1$

3. Place point P on the circumference of a circle with radius 1 such that gen
angle $\theta=-\frac{2}{3}\pi$. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$. ➡ MI

[Sol] The coordinates of point P are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.



$\therefore\ \sin\theta=-\frac{\sqrt{3}}{2}$

$\cos\theta=-\frac{1}{2}$

$\tan\theta=\sqrt{3}$

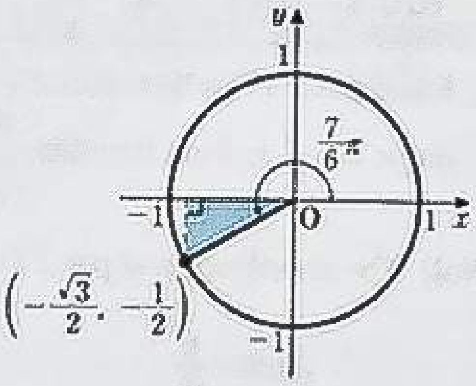
M120b

4. Given $\theta=\frac{11}{12}\pi$, find the value of $\sin\left(\theta+\frac{\pi}{4}\right)$.

➡ M115

[Sol] Since $\frac{11}{12}\pi+\frac{\pi}{4}=\frac{7}{6}\pi$,

$$\sin\left(\theta+\frac{\pi}{4}\right)=-\frac{1}{2}$$

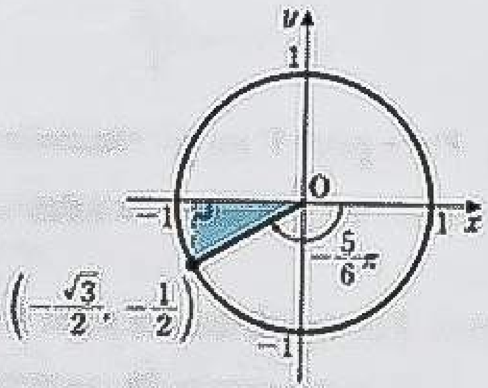


5. Given $\theta=-\frac{\pi}{4}$, find the value of $\cos\left(2\theta-\frac{\pi}{3}\right)$.

➡ M116

[Sol] Since $2\cdot\left(-\frac{\pi}{4}\right)-\frac{\pi}{3}=-\frac{5}{6}\pi$,

$$\cos\left(2\theta-\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$$



6. Given that θ is an angle in the 1st Quadrant and $\cos\theta=\frac{5}{13}$, evaluate the following expressions using the diagram below.

➡ M118

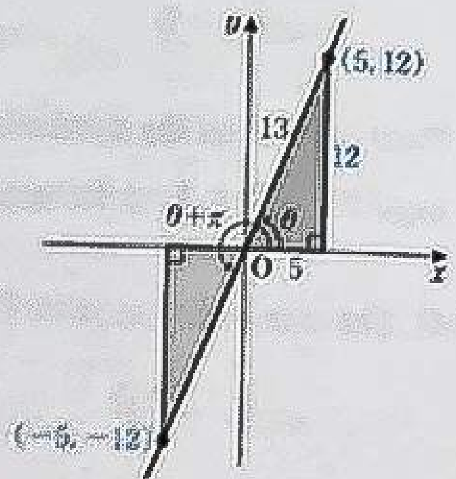
(1) $\sin\theta=\frac{12}{13}$

(2) $\tan\theta=\frac{12}{5}$

(3) $\sin(\theta+\pi)=-\frac{12}{13}$

(4) $\cos(\theta+\pi)=-\frac{5}{13}$

(5) $\tan(\theta+\pi)=\frac{12}{5}$



M121a

KUMON

Trigonometric Equations

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0	—	1	2	3~

Given $0 \leqq \theta < 2\pi$, solve the following equations.

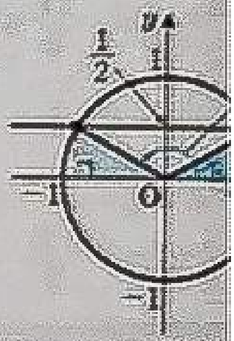
Ex. $2\sin\theta - 1 = 0$

[Sol] Rearranging,

$$\sin\theta = \frac{1}{2}$$

Therefore,

$$\theta = \frac{\pi}{6},\ \frac{5}{6}\pi$$



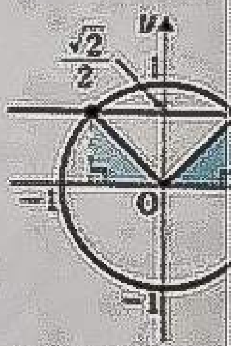
(1) $2\sin\theta - \sqrt{2} = 0$

[Sol] Rearranging,

$$\sin\theta = \frac{\sqrt{2}}{2}$$

Therefore,

$$\theta = \frac{\pi}{4},\ \frac{3}{4}\pi$$



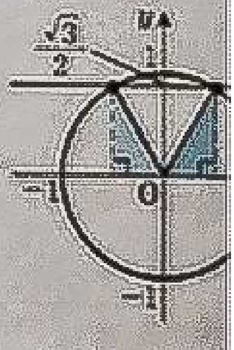
(2) $2\sin\theta - \sqrt{3} = 0$

[Sol] Rearranging,

$$\sin\theta = \frac{\sqrt{3}}{2}$$

Therefore,

$$\theta = \frac{\pi}{3},\ \frac{2}{3}\pi$$



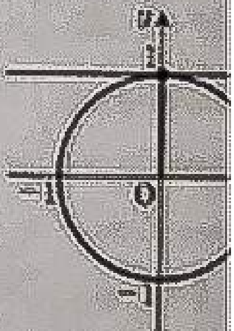
(3) $\sin\theta - 1 = 0$

[Sol] Rearranging,

$$\sin\theta = 1$$

Therefore,

$$\theta = \frac{\pi}{2}$$



M121b

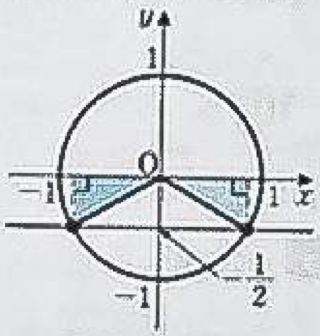
(4) $2\sin\theta+1=0$

[Sol] Rearranging,

$$\sin\theta=-\frac{1}{2}$$

Therefore,

$$\theta=\frac{7}{6}\pi,\ \frac{11}{6}\pi$$



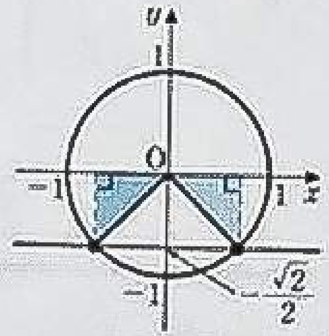
(5) $2\sin\theta+\sqrt{2}=0$

[Sol] Rearranging,

$$\sin\theta=-\frac{\sqrt{2}}{2}$$

Therefore,

$$\theta=\frac{5}{4}\pi,\ \frac{7}{4}\pi$$



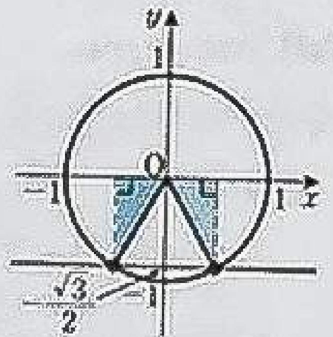
(6) $2\sqrt{3}\sin\theta+3=0$

[Sol] Rearranging,

$$\sin\theta=-\frac{\sqrt{3}}{2}$$

Therefore,

$$\theta=\frac{4}{3}\pi,\ \frac{5}{3}\pi$$



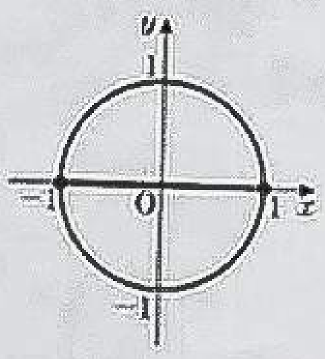
(7) $3\sin\theta=0$

[Sol] Rearranging,

$$\sin\theta=0$$

Therefore,

$$\theta=0,\ \pi$$



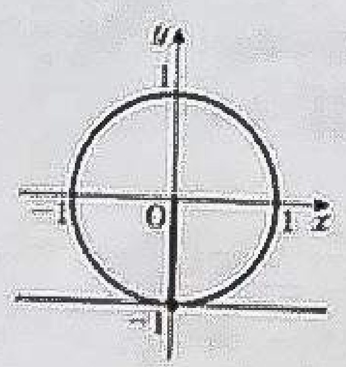
(8) $\sin\theta+1=0$

[Sol] Rearranging,

$$\sin\theta=-1$$

Therefore,

$$\theta=\frac{3}{2}\pi$$



M122a

KUMON

Trigonometric Equations

Name

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Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	2	3~

Given $0 \leqq \theta < 2\pi$, solve the following equations.

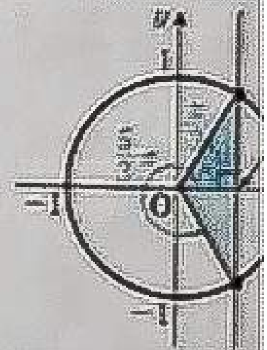
Ex. $2\cos\theta - 1 = 0$

[Sol] Rearranging,

$$\cos\theta = \frac{1}{2}$$

Therefore,

$$\theta = \frac{\pi}{3},\ \frac{5}{3}\pi$$



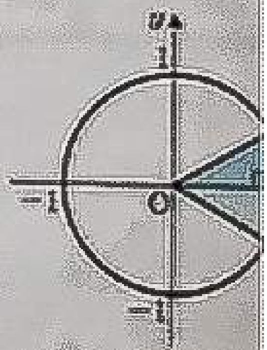
(1) $2\cos\theta - \sqrt{3} = 0$

[Sol] Rearranging,

$$\cos\theta = \frac{\sqrt{3}}{2}$$

Therefore,

$$\theta = \frac{\pi}{6},\ \frac{11}{6}\pi$$



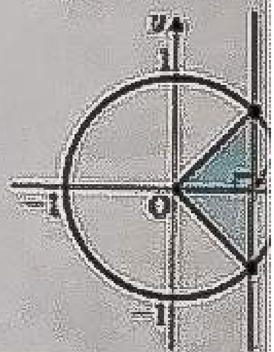
(2) $2\cos\theta - \sqrt{2} = 0$

[Sol] Rearranging,

$$\cos\theta = \frac{\sqrt{2}}{2}$$

Therefore,

$$\theta = \frac{\pi}{4},\ \frac{7}{4}\pi$$



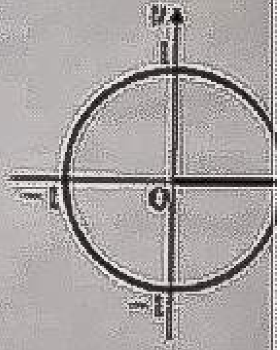
(3) $\cos\theta - 1 = 0$

[Sol] Rearranging,

$$\cos\theta = 1$$

Therefore,

$$\theta = 0$$



M122b

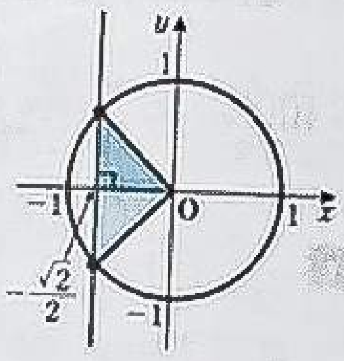
(4) $2\cos\theta+\sqrt{2}=0$

[Sol] Rearranging,

$$\cos\theta=-\frac{\sqrt{2}}{2}$$

Therefore,

$$\theta=\frac{3}{4}\pi,\ \frac{5}{4}\pi$$



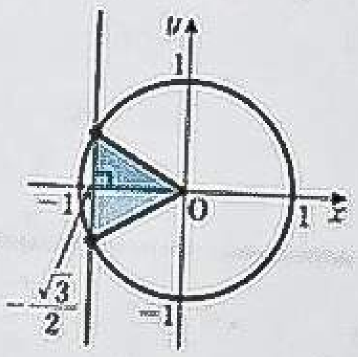
(5) $2\cos\theta+\sqrt{3}=0$

[Sol] Rearranging,

$$\cos\theta=-\frac{\sqrt{3}}{2}$$

Therefore,

$$\theta=\frac{5}{6}\pi,\ \frac{7}{6}\pi$$



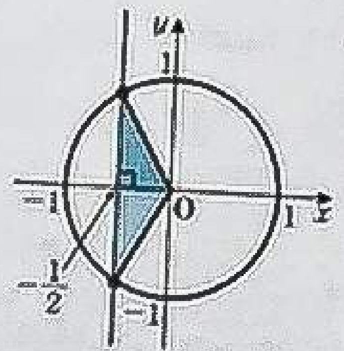
(6) $2\cos\theta+1=0$

[Sol] Rearranging,

$$\cos\theta=-\frac{1}{2}$$

Therefore,

$$\theta=\frac{2}{3}\pi,\ \frac{4}{3}\pi$$



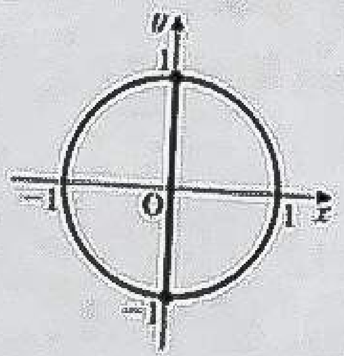
(7) $3\cos\theta=0$

[Sol] Rearranging,

$$\cos\theta=0$$

Therefore,

$$\theta=\frac{\pi}{2},\ \frac{3}{2}\pi$$



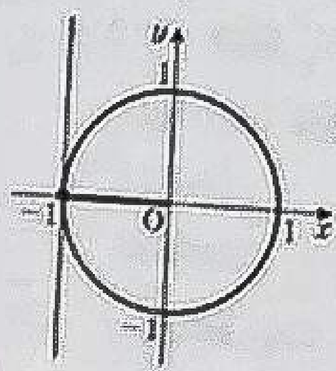
(8) $\cos\theta+1=0$

[Sol] Rearranging,

$$\cos\theta=-1$$

Therefore,

$$\theta=\pi$$



M I 23a

KUMON

Trigonometric Equations

Name

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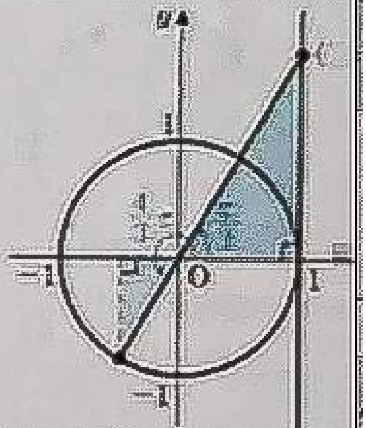
Time : to :

100%	~90%	~80%	~70%	69%~
0		1		2

Given $0 \leqq \theta < 2\pi$, solve the following equations.

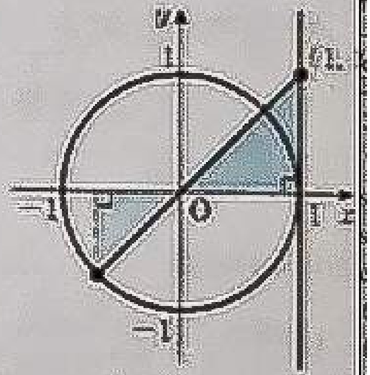
Ex. $\tan\theta = \sqrt{3}$

[Sol] $\theta = \frac{\pi}{3}, \frac{4}{3}\pi$



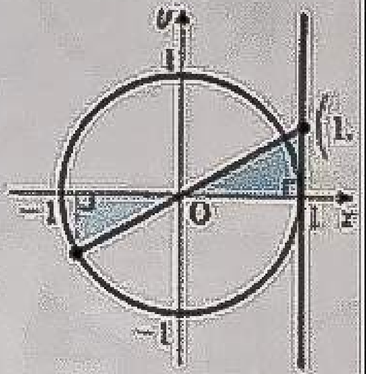
(1) $\tan\theta = 1$

[Sol] $\theta = \frac{\pi}{4}, \frac{5}{4}\pi$



(2) $\tan\theta = \frac{\sqrt{3}}{3}$

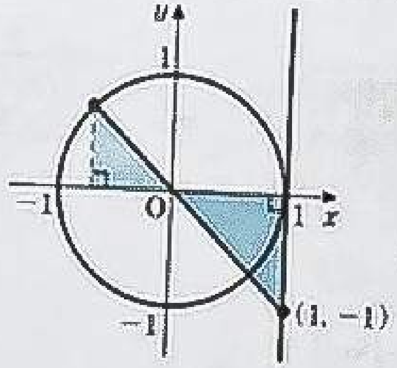
[Sol] $\theta = \frac{\pi}{6}, \frac{7}{6}\pi$



M123b

(3) $\tan\theta=-1$

[Sol] $\theta=\frac{3}{4}\pi,\ \frac{7}{4}\pi$



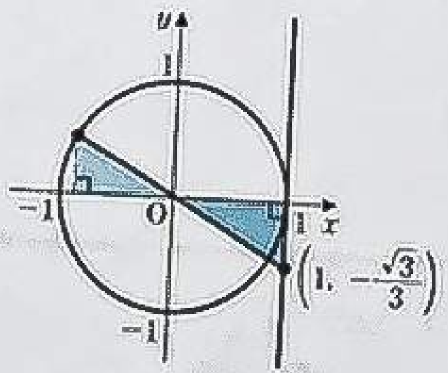
(4) $3\tan\theta+\sqrt{3}=0$

[Sol] Rearranging,

$$\tan\theta=-\frac{\sqrt{3}}{3}$$

Therefore,

$$\theta=\frac{5}{6}\pi,\ \frac{11}{6}\pi$$



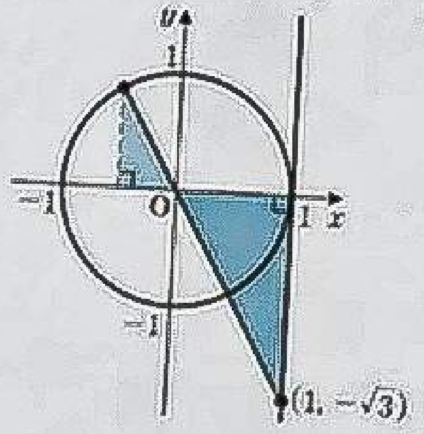
(5) $\sqrt{3}\tan\theta+3=0$

[Sol] Rearranging,

$$\tan\theta=-\sqrt{3}$$

Therefore,

$$\theta=\frac{2}{3}\pi,\ \frac{5}{3}\pi$$



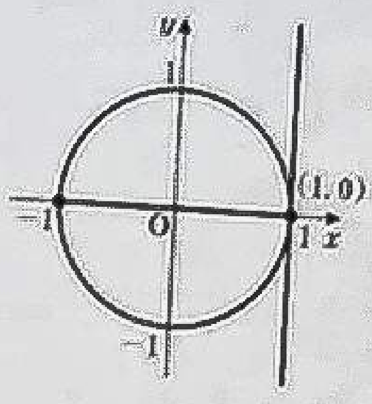
(6) $2\tan\theta=0$

[Sol] Rearranging,

$$\tan\theta=0$$

Therefore,

$$\theta=0,\ \pi$$



M I 24a

KUMON

Trigonometric Equations

Name

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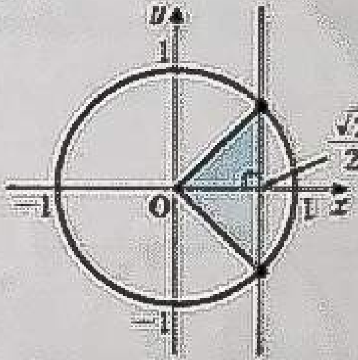
(mistakes) 0	—	1	2	3~

Solve the following equations.

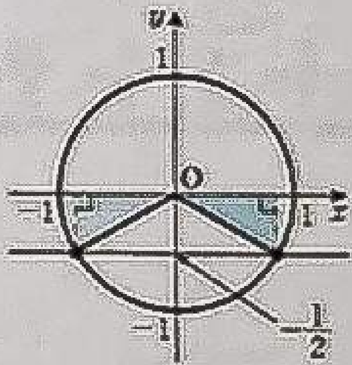
(1) $\cos\theta=\frac{\sqrt{2}}{2}$ $(-\pi \leqq \theta < \pi)$

[Sol] $\theta=-\frac{\pi}{4}, \frac{\pi}{4}$



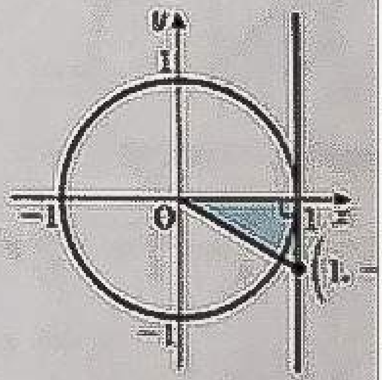
(2) $\sin\theta=-\frac{1}{2}$ $(-\pi \leqq \theta < \pi)$

[Sol] $\theta=-\frac{5}{6}\pi, -\frac{\pi}{6}$



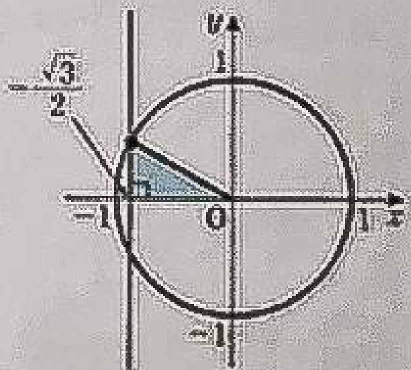
(3) $\tan\theta=-\frac{\sqrt{3}}{3}$ $\left(-\frac{\pi}{2} \leqq \theta < \frac{\pi}{2}\right)$

[Sol] $\theta=-\frac{\pi}{6}$ ⟵ Since $-\frac{\pi}{2} \leqq \theta < \frac{\pi}{2}$, $\theta=\frac{5}{6}\pi$ is not included.



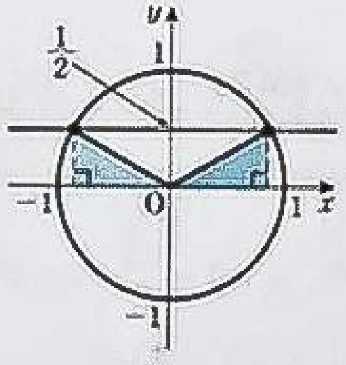
(4) $\cos\theta=-\frac{\sqrt{3}}{2}$ $(0 \leqq \theta < \pi)$

[Sol] $\theta=\frac{5}{6}\pi$ ⟵ Since $0 \leqq \theta < \pi$, $\theta=\frac{7}{6}\pi$ is not included.



M124b

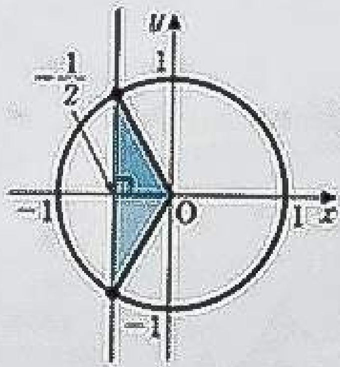
(5) $\sin\theta = \frac{1}{2}$ $(0 \leqq \theta < 4\pi)$



[Sol] $\theta = \boxed{\frac{\pi}{6}}, \boxed{\frac{5}{6}\pi}, \boxed{\frac{13}{6}\pi}, \boxed{\frac{17}{6}\pi}$

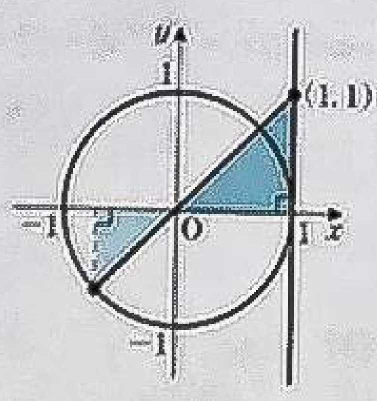
(6) $\cos\theta = -\frac{1}{2}$ $(0 \leqq \theta < 4\pi)$

[Sol] $\theta = \frac{2}{3}\pi, \frac{4}{3}\pi, \frac{8}{3}\pi, \frac{10}{3}\pi$



(7) $\tan\theta = 1$ $(0 \leqq \theta < 4\pi)$

[Sol] $\theta = \frac{\pi}{4}, \frac{5}{4}\pi, \frac{9}{4}\pi, \frac{13}{4}\pi$



Trigonometric Equations

Name

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Given $0 \leqq \theta < 2\pi$, solve the following equations.

Ex.

$\sin 2\theta = \frac{\sqrt{3}}{2}$

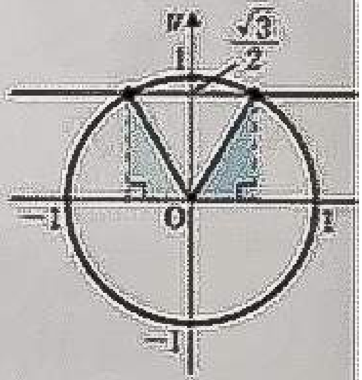
[Sol] Since $0 \leqq \theta < 2\pi$,

$0 \leqq 2\theta < 4\pi$

Therefore,

$2\theta = \frac{\pi}{3}, \frac{2}{3}\pi, \frac{7}{3}\pi, \frac{8}{3}\pi$

$\therefore \theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7}{6}\pi, \frac{4}{3}\pi$



(1) $\cos 2\theta = \frac{\sqrt{2}}{2}$

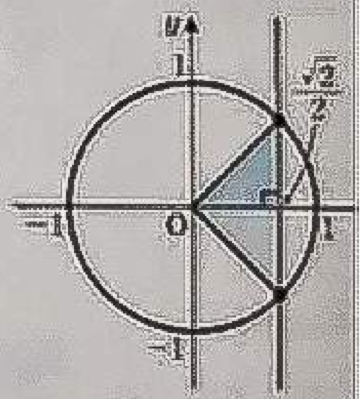
[Sol] Since $0 \leqq \theta < 2\pi$,

$0 \leqq 2\theta < 4\pi$

Therefore,

$2\theta = \frac{\pi}{4}, \frac{7}{4}\pi, \frac{9}{4}\pi, \frac{15}{4}\pi$

$\therefore \theta = \frac{\pi}{8}, \frac{7}{8}\pi, \frac{9}{8}\pi, \frac{15}{8}\pi$



(2) $\tan 2\theta = \frac{\sqrt{3}}{3}$

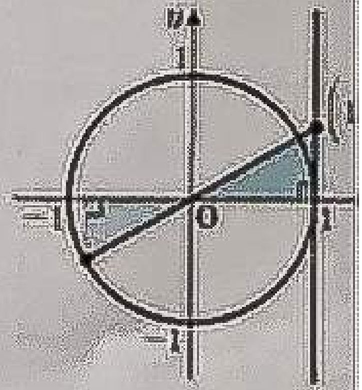
[Sol] Since $0 \leqq \theta < 2\pi$,

$0 \leqq 2\theta < 4\pi$

Therefore,

$2\theta = \frac{\pi}{6}, \frac{7}{6}\pi, \frac{13}{6}\pi, \frac{19}{6}\pi$

$\therefore \theta = \frac{\pi}{12}, \frac{7}{12}\pi, \frac{13}{12}\pi, \frac{19}{12}\pi$



M125b

(3) $\sin\frac{\theta}{2}=\frac{1}{2}$

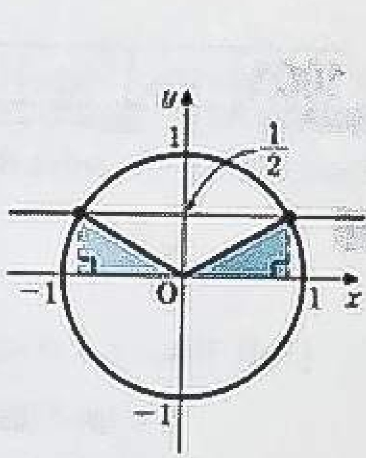
[Sol] Since $0 \leqq \theta < 2\pi$,

$$0 \leqq \frac{\theta}{2} < \pi$$

Therefore,

$$\frac{\theta}{2}=\frac{\pi}{6},\ \frac{5}{6}\pi$$

$$\therefore\ \boldsymbol{\theta=\frac{\pi}{3},\ \frac{5}{3}\pi}$$



(4) $2\cos\frac{\theta}{2}+\sqrt{2}=0$

[Sol] Rearranging,

$$\cos\frac{\theta}{2}=-\frac{\sqrt{2}}{2}$$

Since $0 \leqq \theta < 2\pi$,

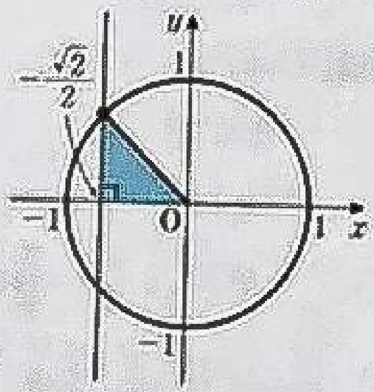
$$0 \leqq \frac{\theta}{2} < \pi$$

Therefore,

$$\frac{\theta}{2}=\frac{3}{4}\pi$$

Since $0 \leqq \frac{\theta}{2} < \pi$, $\frac{\theta}{2}=\frac{5}{4}\pi$ is not included.

$$\therefore\ \boldsymbol{\theta=\frac{3}{2}\pi}$$



M I 26a KUMON

Trigonometric Equations

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Given $0 \leqq \theta < 2\pi$, solve the following equations.

Ex. $\sin\left(\theta+\frac{\pi}{3}\right)=\frac{1}{2}$

[Sol] Since $0 \leqq \theta < 2\pi$,

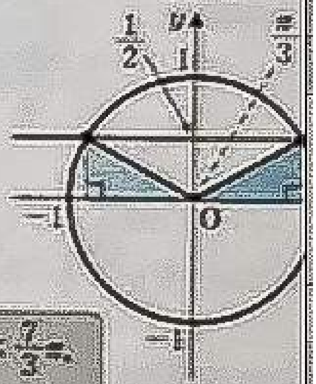
$$\frac{\pi}{3} \leqq \theta+\frac{\pi}{3} < \frac{7}{3}\pi$$

Therefore,

$$\theta+\frac{\pi}{3}=\frac{5}{6}\pi,\ \frac{13}{6}\pi$$

Since $\frac{\pi}{3} \leqq \theta+\frac{\pi}{3} < \frac{7}{3}\pi$, $\theta+\frac{\pi}{3}=\frac{\pi}{6}$ is not included.

$$\therefore\ \theta=\frac{\pi}{2},\ \frac{11}{6}\pi$$



(1) $\cos\left(\theta+\frac{\pi}{4}\right)=\frac{\sqrt{3}}{2}$

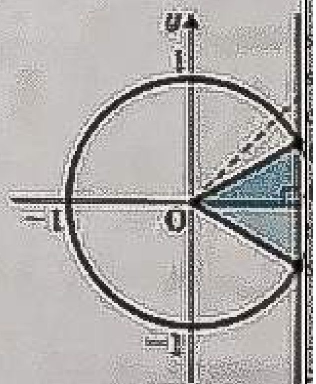
[Sol] Since $0 \leqq \theta < 2\pi$,

$$\frac{\pi}{4} \leqq \theta+\frac{\pi}{4} < \frac{9}{4}\pi$$

Therefore,

$$\theta+\frac{\pi}{4}=\frac{11}{6}\pi,\ \frac{13}{6}\pi$$

$$\therefore\ \theta=\frac{19}{12}\pi,\ \frac{23}{12}\pi$$



(2) $\tan\left(\theta+\frac{\pi}{6}\right)=1$

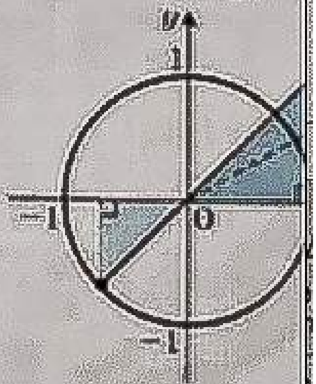
[Sol] Since $0 \leqq \theta < 2\pi$,

$$\frac{\pi}{6} \leqq \theta+\frac{\pi}{6} < \frac{13}{6}\pi$$

Therefore,

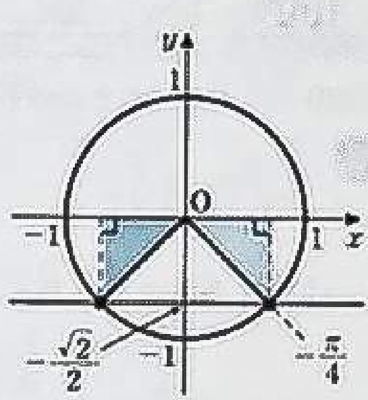
$$\theta+\frac{\pi}{6}=\frac{\pi}{4},\ \frac{5}{4}\pi$$

$$\therefore\ \theta=\frac{\pi}{12},\ \frac{13}{12}\pi$$



M126b

(3) $\sin\left(\theta-\dfrac{\pi}{4}\right)=-\dfrac{\sqrt{2}}{2}$



[Sol] Since $0 \leqq \theta < 2\pi$,

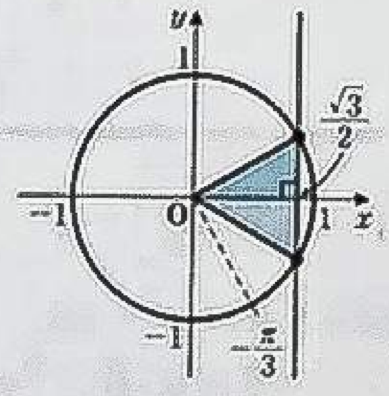
$$-\frac{\pi}{4} \leqq \theta-\frac{\pi}{4} < \frac{7}{4}\pi$$

Therefore,

$$\theta-\frac{\pi}{4}=-\frac{\pi}{4},\ \frac{5}{4}\pi$$

$$\therefore\ \theta=0,\ \frac{3}{2}\pi$$

(4) $2\sqrt{3}\cos\left(\theta-\dfrac{\pi}{3}\right)-3=0$



[Sol] Rearranging,

$$\cos\left(\theta-\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

Since $0 \leqq \theta < 2\pi$,

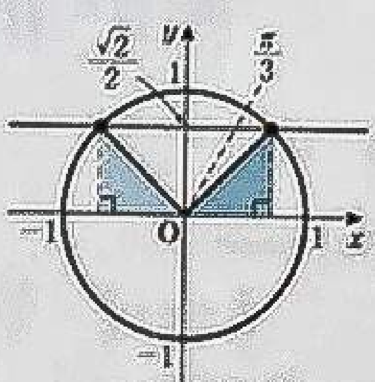
$$-\frac{\pi}{3} \leqq \theta-\frac{\pi}{3} < \frac{5}{3}\pi$$

Therefore,

$$\theta-\frac{\pi}{3}=-\frac{\pi}{6},\ \frac{\pi}{6}$$

$$\therefore\ \theta=\frac{\pi}{6},\ \frac{\pi}{2}$$

(5) $\sqrt{2}\sin\left(2\theta+\dfrac{\pi}{3}\right)-1=0$



[Sol] Rearranging,

$$\sin\left(2\theta+\frac{\pi}{3}\right)=\frac{\sqrt{2}}{2}$$

Since $0 \leqq \theta < 2\pi$,

$$\frac{\pi}{3} \leqq 2\theta+\frac{\pi}{3} < \frac{13}{3}\pi$$

Therefore,

$$2\theta+\frac{\pi}{3}=\frac{3}{4}\pi,\ \frac{9}{4}\pi,\ \frac{11}{4}\pi,\ \frac{17}{4}\pi$$

$$2\theta=\frac{5}{12}\pi,\ \frac{23}{12}\pi,\ \frac{29}{12}\pi,\ \frac{47}{12}\pi$$

$$\therefore\ \theta=\frac{5}{24}\pi,\ \frac{23}{24}\pi,\ \frac{29}{24}\pi,\ \frac{47}{24}\pi$$

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Trigonometric Equations

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Given $0 \leqq \theta < 2\pi$, solve the following equations.

Ex. $2\sin^2\theta+3\sin\theta-2=0$

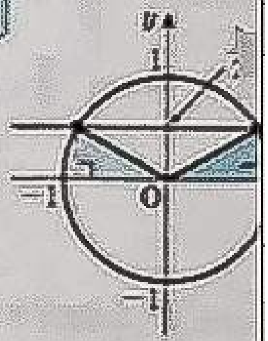
[Sol] $(2\sin\theta-1)(\sin\theta+2)=0$ ← $2x^2+3x-2=(2x-1)(x+2)$

$\sin\theta=\frac{1}{2},\ -2$

Since $-1 \leqq \sin\theta \leqq 1$, ← $0 \leqq \theta < 2\pi$

$\sin\theta=\frac{1}{2}$

$\therefore\ \theta=\frac{\pi}{6},\ \frac{5}{6}\pi$



(1) $2\cos^2\theta-5\cos\theta+2=0$

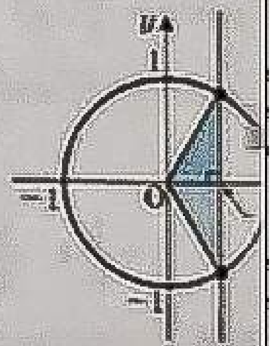
[Sol] $(2\cos\theta-1)(\cos\theta-2)=0$

$\cos\theta=\frac{1}{2},\ 2$

Since $-1 \leqq \cos\theta \leqq 1$,

$\cos\theta=\frac{1}{2}$

$\therefore\ \theta=\frac{\pi}{3},\ \frac{5}{3}\pi$



(2) $\sqrt{3}\sin\theta-2\sin^2\theta=0$

[Sol] $\sin\theta(\sqrt{3}-2\sin\theta)=0$

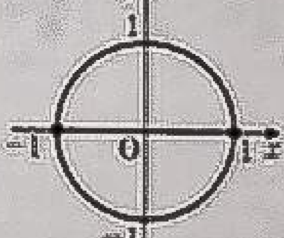
$\sin\theta=0,\ \frac{\sqrt{3}}{2}$ ← Both values are included in the range $-1 \leqq \sin\theta \leqq 1$.

$\therefore\ \theta=0,\ \frac{\pi}{3},\ \frac{2}{3}\pi,\ \pi$

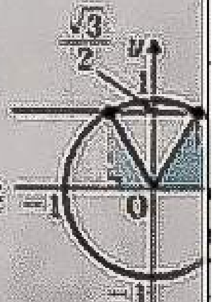
When $\sin\theta=0$, $\theta=0,\ \pi$

When $\sin\theta=\frac{\sqrt{3}}{2}$, $\theta=\frac{\pi}{3},\ \frac{2}{3}\pi$

(When $\sin\theta=0$)



(When $\sin\theta=$



M127b

(3) $2\sin^2\theta+\cos\theta-1=0$

[Sol] $2(1-\cos^2\theta)+\cos\theta-1=0$ $\quad$ $\boxed{\sin^2\theta=1-\cos^2\theta}$

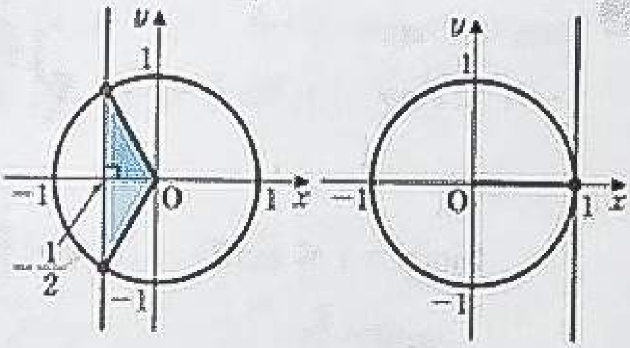
$2\cos^2\theta-\cos\theta-1=0$

$(2\cos\theta+1)(\cos\theta-1)=0$

$\cos\theta=-\dfrac{1}{2},\ 1$

$\therefore\ \theta=0,\ \dfrac{2}{3}\pi,\ \dfrac{4}{3}\pi$

[When $\cos\theta=-\dfrac{1}{2}$] $\quad$ [When $\cos\theta=1$]



(4) $2\cos^2\theta+5\sin\theta+1=0$

[Sol] $2(1-\sin^2\theta)+5\sin\theta+1=0$

$2\sin^2\theta-5\sin\theta-3=0$

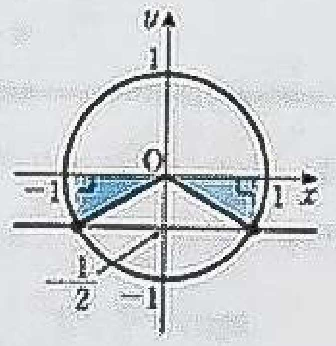
$(2\sin\theta+1)(\sin\theta-3)=0$

$\sin\theta=-\dfrac{1}{2},\ 3$

Since $-1\leqq\sin\theta\leqq1$,

$\sin\theta=-\dfrac{1}{2}$

$\therefore\ \theta=\dfrac{7}{6}\pi,\ \dfrac{11}{6}\pi$



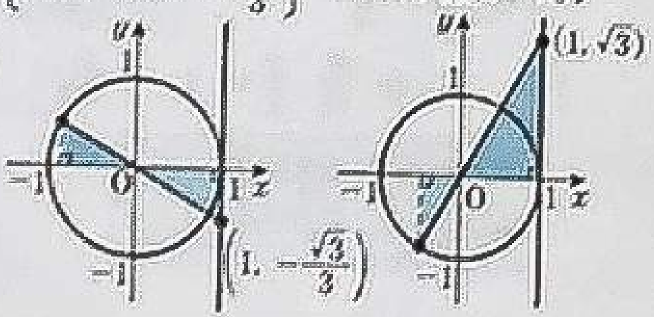
(5) $3\tan^2\theta-2\sqrt{3}\tan\theta-3=0$

[Sol] $(\boxed{3\tan\theta+\sqrt{3}})(\boxed{\tan\theta-\sqrt{3}})=0$ $\quad$ $[(\sqrt{3}\tan\theta+1)(\sqrt{3}\tan\theta-3)=0]$

$\tan\theta=\boxed{-\dfrac{\sqrt{3}}{3}},\ \boxed{\sqrt{3}}$ $\quad$ For $0\leqq\theta<2\pi$, $\tan\theta$ takes all real values.

$\therefore\ \theta=\boxed{\dfrac{\pi}{3}},\ \boxed{\dfrac{5}{6}\pi},\ \boxed{\dfrac{4}{3}\pi},\ \boxed{\dfrac{11}{6}\pi}$

[When $\tan\theta=-\dfrac{\sqrt{3}}{3}$] $\quad$ [When $\tan\theta=\sqrt{3}$]



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Trigonometric Equations

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Given $0 \leqq \theta < 2\pi$, solve the following equations.

(1) $2\cos\theta - \sqrt{2}\tan\theta = 0$

[Sol] Since $2\cos\theta - \sqrt{2}\cdot\dfrac{\sin\theta}{\cos\theta} = 0$, $\boxed{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$

$$2\cos^2\theta - \sqrt{2}\sin\theta = 0$$

$$2(1-\sin^2\theta) - \sqrt{2}\sin\theta = 0 \qquad \boxed{\cos^2\theta = 1-\sin^2\theta}$$

$$2\sin^2\theta + \sqrt{2}\sin\theta - 2 = 0$$

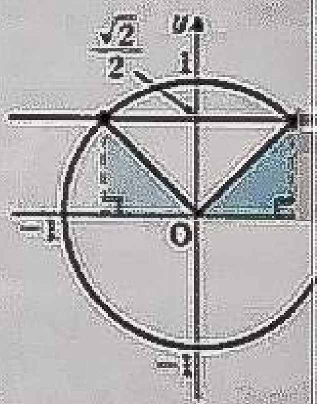
$$(\sin\theta + \sqrt{2})(2\sin\theta - \sqrt{2}) = 0$$

$$\sin\theta = -\sqrt{2},\ \frac{\sqrt{2}}{2}$$

Since $-1 \leqq \sin\theta \leqq 1$,

$$\sin\theta = \frac{\sqrt{2}}{2}$$

$$\therefore\ \theta = \frac{\pi}{4},\ \frac{3}{4}\pi$$



(2) $2\sqrt{3}\cos\theta + \tan\theta = 0$

[Sol] Since $2\sqrt{3}\cos\theta + \dfrac{\sin\theta}{\cos\theta} = 0$,

$$2\sqrt{3}\cos^2\theta + \sin\theta = 0$$

$$2\sqrt{3}(1-\sin^2\theta) + \sin\theta = 0$$

$$2\sqrt{3}\sin^2\theta - \sin\theta - 2\sqrt{3} = 0$$

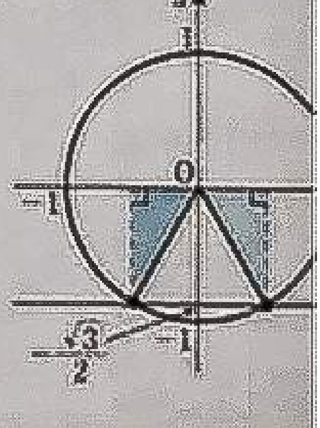
$$(2\sin\theta + \sqrt{3})(\sqrt{3}\sin\theta - 2) = 0$$

$$\sin\theta = -\frac{\sqrt{3}}{2},\ \frac{2\sqrt{3}}{3}$$

Since $-1 \leqq \sin\theta \leqq 1$,

$$\sin\theta = -\frac{\sqrt{3}}{2}$$

$$\therefore\ \theta = \frac{4}{3}\pi,\ \frac{5}{3}\pi$$



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(3) $2\sin\theta\tan\theta=-3$

[Sol] Since $2\sin\theta\cdot\dfrac{\sin\theta}{\cos\theta}=-3$,

$$2\sin^2\theta+3\cos\theta=0$$

$$2(1-\cos^2\theta)+3\cos\theta=0$$

$$2\cos^2\theta-3\cos\theta-2=0$$

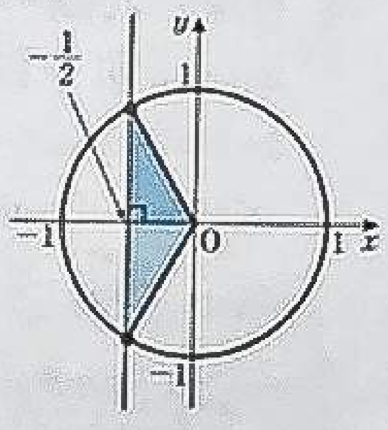
$$(2\cos\theta+1)(\cos\theta-2)=0$$

$$\cos\theta=-\frac{1}{2},\ 2$$

Since $-1\leqq\cos\theta\leqq1$,

$$\cos\theta=-\frac{1}{2}$$

$$\therefore\ \theta=\frac{2}{3}\pi,\ \frac{4}{3}\pi$$



(4) $2\sin\theta=\tan\theta$

[Sol] Since $2\sin\theta=\dfrac{\sin\theta}{\cos\theta}$,

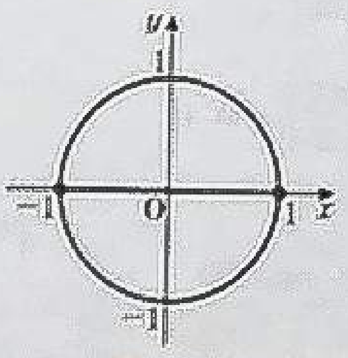
$$2\sin\theta\cos\theta-\sin\theta=0$$

$$\sin\theta(2\cos\theta-1)=0$$

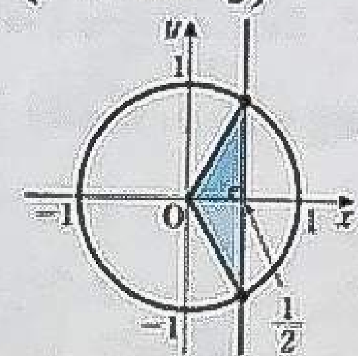
$$\sin\theta=0,\ \cos\theta=\frac{1}{2}$$

$$\therefore\ \theta=0,\ \frac{\pi}{3},\ \pi,\ \frac{5}{3}\pi$$

[When $\sin\theta=0$]



[When $\cos\theta=\frac{1}{2}$]



When $\sin\theta=0$, $\theta=0,\ \pi$
When $\cos\theta=\frac{1}{2}$, $\theta=\frac{\pi}{3},\ \frac{5}{3}\pi$

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KUMON

Trigonometric Equations

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1. Given $0 \leqq \theta < 2\pi$, solve the equation $\cos^2\theta + \sqrt{3}\sin\theta\cos\theta = 1$.

[Sol] Since $(1-\sin^2\theta) + \sqrt{3}\sin\theta\cos\theta = 1$, ⟵ $\boxed{\cos^2\theta = 1-\sin^2\theta}$

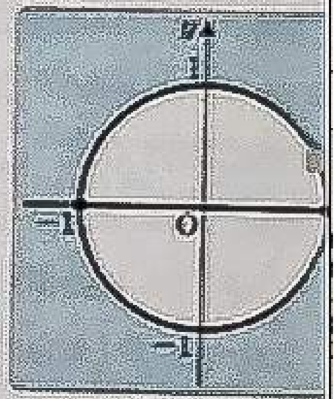
$$\sin^2\theta - \sqrt{3}\sin\theta\cos\theta = 0$$

$$\sin\theta(\sin\theta - \sqrt{3}\cos\theta) = 0$$

$$\therefore\ \sin\theta = 0,\ \sin\theta - \sqrt{3}\cos\theta = 0$$

(i) When $\sin\theta = 0$,

$$\theta = 0,\ \pi$$



(ii) When $\sin\theta - \sqrt{3}\cos\theta = 0$,

$$\sin\theta = \sqrt{3}\cos\theta \quad \cdots ①$$

$\boxed{\text{Since } \sin^2\theta + \cos^2\theta = 1}$

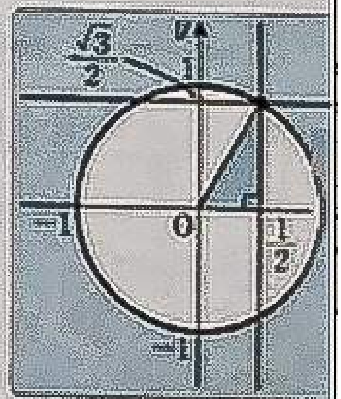
$$\therefore\ (\sqrt{3}\cos\theta)^2 + \cos^2\theta = 1$$

$$\cos^2\theta = \frac{1}{4}$$

$$\therefore\ \cos\theta = \pm\frac{1}{2}$$

When $\cos\theta = \frac{1}{2}$, from ①, $\sin\theta = \frac{\sqrt{3}}{2}$

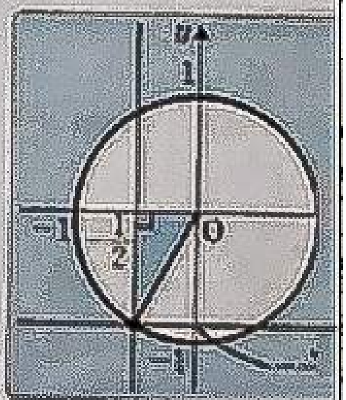
$$\therefore\ \theta = \frac{\pi}{3}$$



When $\cos\theta = -\frac{1}{2}$, from ①, $\sin\theta = -\frac{\sqrt{3}}{2}$

$$\therefore\ \theta = \frac{4}{3}\pi$$

$$\therefore\ \theta = \frac{\pi}{3},\ \frac{4}{3}\pi$$



From (i) and (ii), $\theta = 0,\ \frac{\pi}{3},\ \pi,\ \frac{4}{3}\pi$

M129b

2. Given two points $P(\sin\theta+2,\ \tan\theta-2)$ and $Q(4\sin^2\theta+4\sin\theta\cos\theta+2a\cos\theta,\ 3\sin\theta-2\cos\theta+a)$, find constant a and the corresponding value of θ when these two points coincide. $(0 \leqslant \theta < 2\pi)$

[Sol] When points P and Q coincide,

$$\sin\theta+2=4\sin^2\theta+4\sin\theta\cos\theta+2a\cos\theta \cdots ①$$

$$\tan\theta-2=3\sin\theta-2\cos\theta+a \cdots ②$$

From ②, $a=\tan\theta-3\sin\theta+2\cos\theta-2 \cdots ③$

From ① and ③,

$\tan\theta\cos\theta=\sin\theta$

$$\sin\theta+2=4\sin^2\theta+4\sin\theta\cos\theta+2(\tan\theta-3\sin\theta+2\cos\theta-2)\cos\theta$$

$$\sin\theta+2=4\sin^2\theta+4\sin\theta\cos\theta+2\sin\theta-6\sin\theta\cos\theta+4\cos^2\theta-4\cos\theta$$

$$2\sin\theta\cos\theta-4(\sin^2\theta+\cos^2\theta)-\sin\theta+4\cos\theta+2=0$$

$\sin^2\theta+\cos^2\theta=1$

$$2\sin\theta\cos\theta-\sin\theta+4\cos\theta-2=0$$

$$2\cos\theta(\sin\theta+2)-(\sin\theta+2)=0$$

$$(\sin\theta+2)(2\cos\theta-1)=0$$

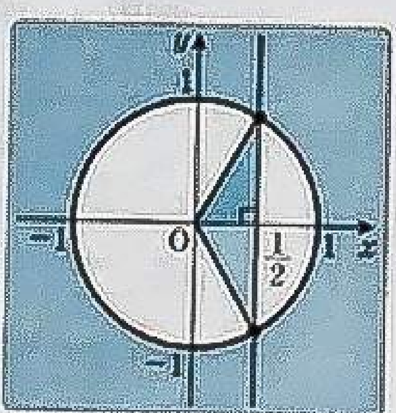
$$\therefore\ \sin\theta=-2,\ \cos\theta=\frac{1}{2}$$

Since $-1 \leqslant \sin\theta \leqslant 1$ and $-1 \leqslant \cos\theta \leqslant 1$,

$$\cos\theta=\frac{1}{2}$$

$0 \leqslant \theta < 2\pi$

$$\therefore\ \theta=\frac{\pi}{3},\ \frac{5}{3}\pi$$



When $\theta=\frac{\pi}{3}$, from ③, $a=-\frac{\sqrt{3}}{2}-1$

$a=\tan\frac{\pi}{3}-3\sin\frac{\pi}{3}+2\cos\frac{\pi}{3}-2$

When $\theta=\frac{5}{3}\pi$, from ③, $a=\frac{\sqrt{3}}{2}-1$

$a=\tan\frac{5}{3}\pi-3\sin\frac{5}{3}\pi+2\cos\frac{5}{3}\pi-2$

Therefore, when $a=-\frac{\sqrt{3}}{2}-1,\ \theta=\frac{\pi}{3}$

when $a=\frac{\sqrt{3}}{2}-1,\ \theta=\frac{5}{3}\pi$

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KUMON

Trigonometric Equations

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Given $0 \leqq \theta < 2\pi$, solve the following equations.

(1) $\sin 2\theta = \frac{1}{2}$

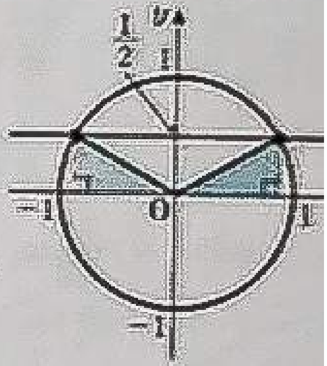
[Sol] Since $0 \leqq \theta < 2\pi$,

$$0 \leqq 2\theta < 4\pi$$

Therefore,

$$2\theta = \frac{\pi}{6},\ \frac{5}{6}\pi,\ \frac{13}{6}\pi,\ \frac{17}{6}\pi$$

$$\therefore\ \theta = \frac{\pi}{12},\ \frac{5}{12}\pi,\ \frac{13}{12}\pi,\ \frac{17}{12}\pi$$



(2) $\cos\frac{\theta}{2} = \frac{\sqrt{2}}{2}$

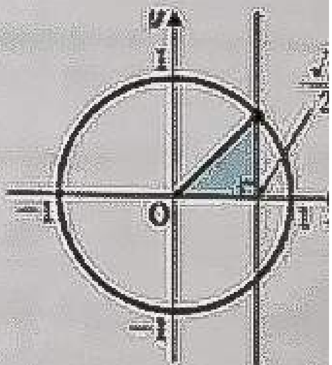
[Sol] Since $0 \leqq \theta < 2\pi$,

$$0 \leqq \frac{\theta}{2} < \pi$$

Therefore,

$$\frac{\theta}{2} = \frac{\pi}{4}$$

$$\therefore\ \theta = \frac{\pi}{2}$$



(3) $\sin\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{2}$

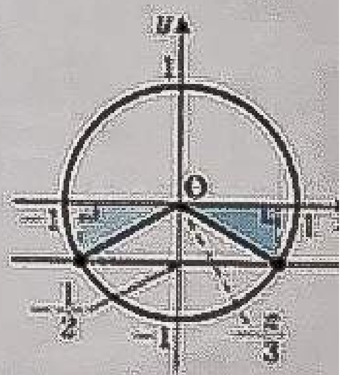
[Sol] Since $0 \leqq \theta < 2\pi$,

$$-\frac{\pi}{3} \leqq \theta - \frac{\pi}{3} < \frac{5}{3}\pi$$

Therefore,

$$\theta - \frac{\pi}{3} = -\frac{\pi}{6},\ \frac{7}{6}\pi$$

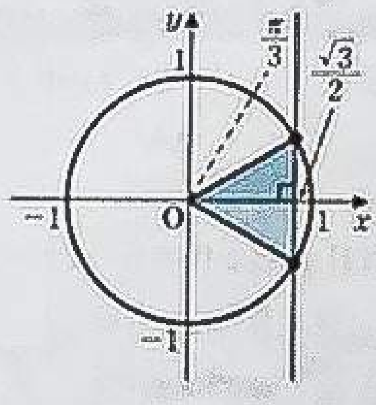
$$\therefore\ \theta = \frac{\pi}{6},\ \frac{3}{2}\pi$$



M130b

(4) $\cos\left(2\theta+\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$

➡ M126



[Sol] Since $0 \leqq \theta < 2\pi$,

$$\frac{\pi}{3} \leqq 2\theta+\frac{\pi}{3} < \frac{13}{3}\pi$$

Therefore,

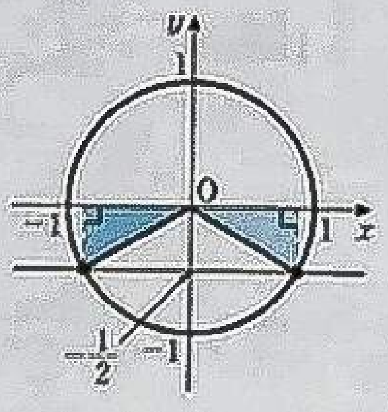
$$2\theta+\frac{\pi}{3}=\frac{11}{6}\pi,\ \frac{13}{6}\pi,\ \frac{23}{6}\pi,\ \frac{25}{6}\pi$$

$$2\theta=\frac{3}{2}\pi,\ \frac{11}{6}\pi,\ \frac{7}{2}\pi,\ \frac{23}{6}\pi$$

$$\therefore\ \theta=\frac{3}{4}\pi,\ \frac{11}{12}\pi,\ \frac{7}{4}\pi,\ \frac{23}{12}\pi$$

(5) $2\cos^2\theta+3\sin\theta=0$

➡ M127



[Sol] $2(1-\sin^2\theta)+3\sin\theta=0$

$$2\sin^2\theta-3\sin\theta-2=0$$

$$(2\sin\theta+1)(\sin\theta-2)=0$$

$$\sin\theta=-\frac{1}{2},\ 2$$

Since $-1 \leqq \sin\theta \leqq 1$,

$$\sin\theta=-\frac{1}{2}$$

$$\therefore\ \theta=\frac{7}{6}\pi,\ \frac{11}{6}\pi$$

M131a

KUMON

Graphs of Trigonometric Functions

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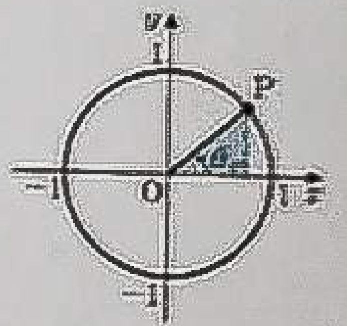
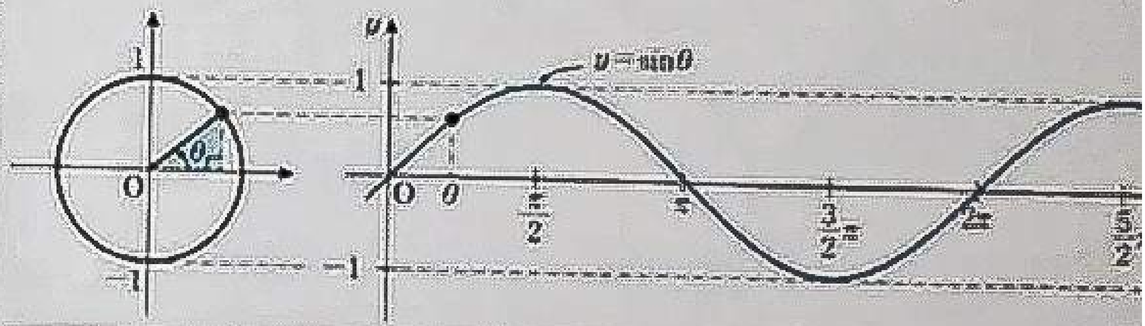
Graph of $y=\sin\theta$

Let the point of intersection of the terminal side of θ and the unit circle be P.
The y-coordinate of P is $\sin\theta$.
Using this, it is possible to draw the graph of $y=\sin\theta$.

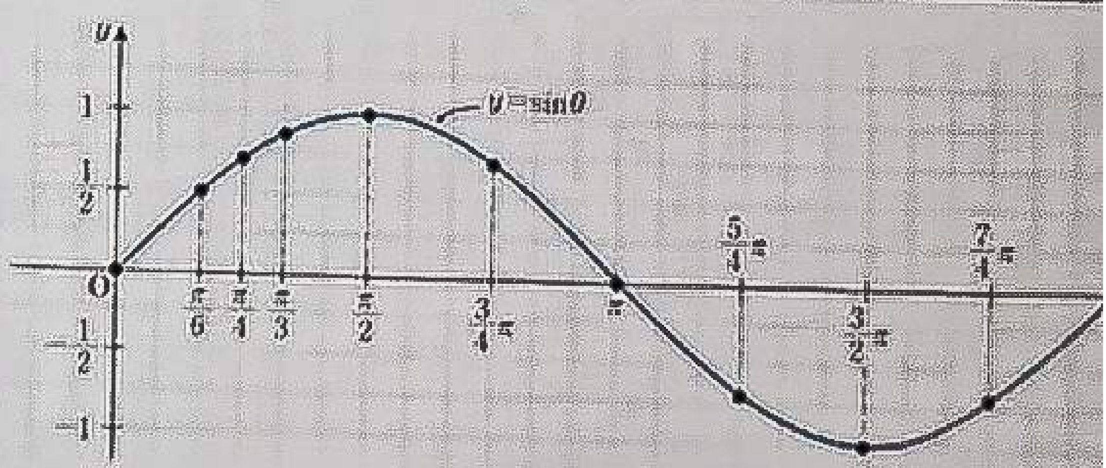




1. Draw the graph of $y=\sin\theta$ for $0 \leqq \theta \leqq 2\pi$.

[Sol]

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$
y	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$



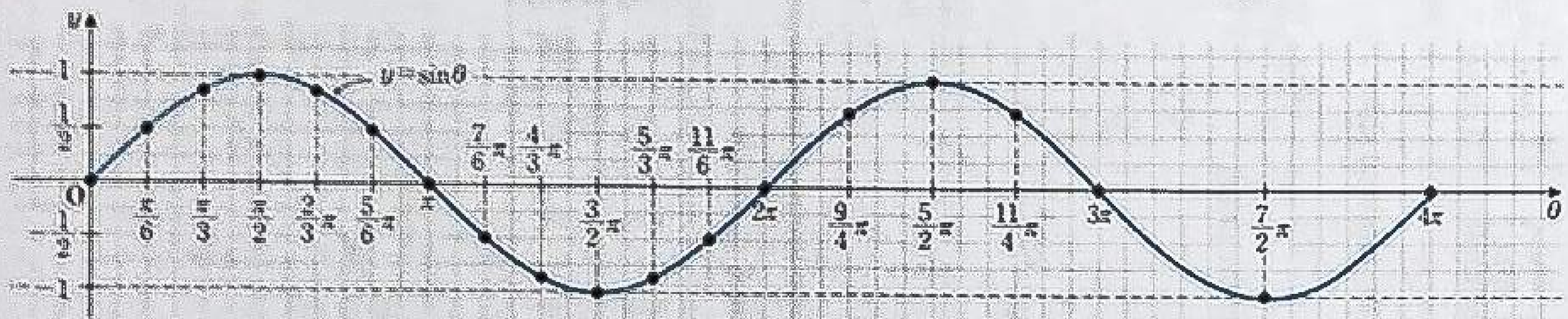
[Note] $\sqrt{2}=1.41\ldots$, $\sqrt{3}=1.73\ldots$, $\pi=3.14\ldots$

M131b

2. Draw the graph of $y = \sin\theta$ for $0 \leqslant \theta \leqslant 4\pi$. Also, fill in the blanks below.

[Sol]

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$	2π	$\frac{9}{4}\pi$	$\frac{5}{2}\pi$	$\frac{11}{4}\pi$	3π	$\frac{7}{2}\pi$	4π
y	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	-1	0



The graph of $y = \sin\theta$ repeats itself every 2π units.
Therefore, 2π is called the ***period*** of this graph.

The period of $y = \sin\theta$ is $\boxed{2\pi}$.

Also, based on the graph and the period above, when θ takes all real values, $\boxed{-1} \leqslant \sin\theta \leqslant \boxed{1}$ is true.

M I 32a

KUMON

Graphs of Trigonometric Functions

Name

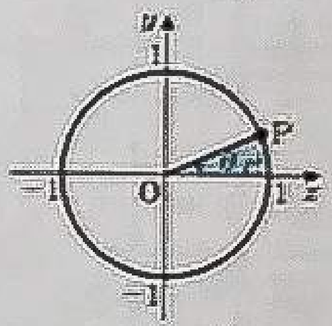
Date / /

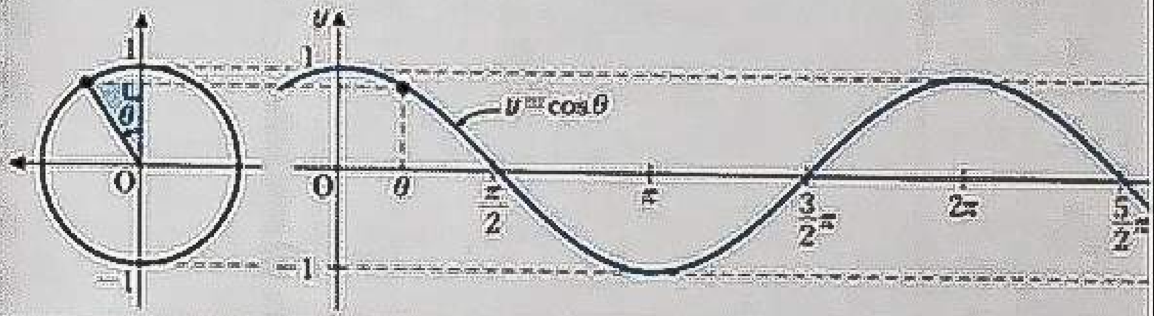
Time : to :

100%	~90%	~80%	~70%	69%~

Graph of $y=\cos\theta$

Let the point of intersection of the terminal side of θ and the unit circle be P.
The x-coordinate of P is $\cos\theta$.
Using this, it is possible to draw the graph of $y=\cos\theta$.

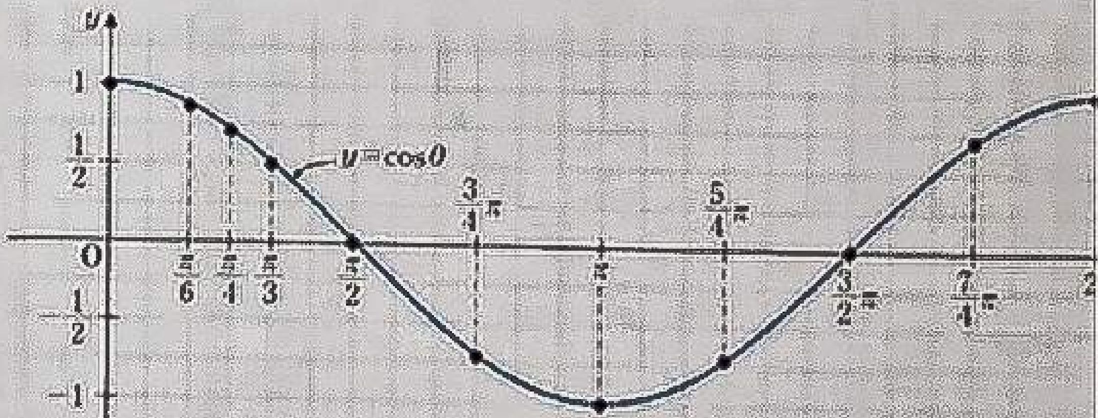




1. Draw the graph of $y=\cos\theta$ for $0 \leqq \theta \leqq 2\pi$.

[Sol]

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$
y	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$



Graphs of Trigonometric Functions

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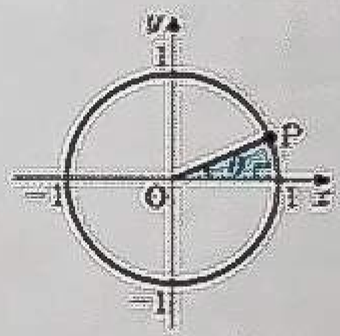
Name

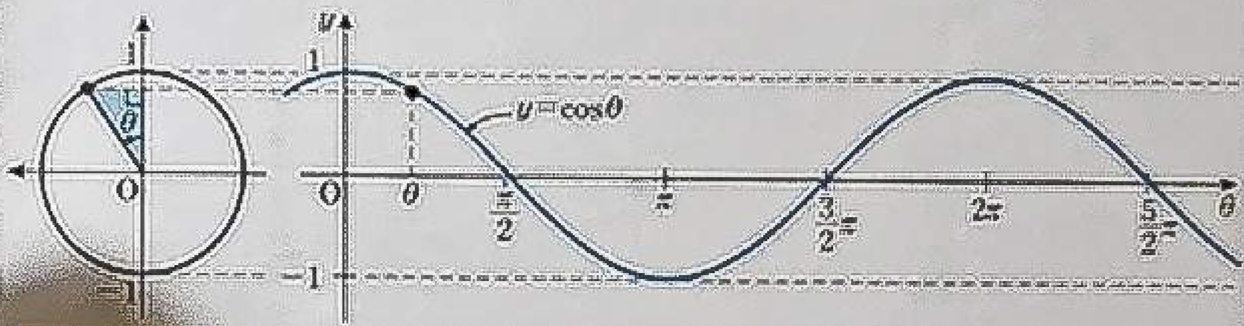
Date / /

Time : to :

Graph of $y = \cos\theta$

Let the point of intersection of the terminal side of θ and the unit circle be P.
The x-coordinate of P is $\cos\theta$.
Using this, it is possible to draw the graph of $y = \cos\theta$.

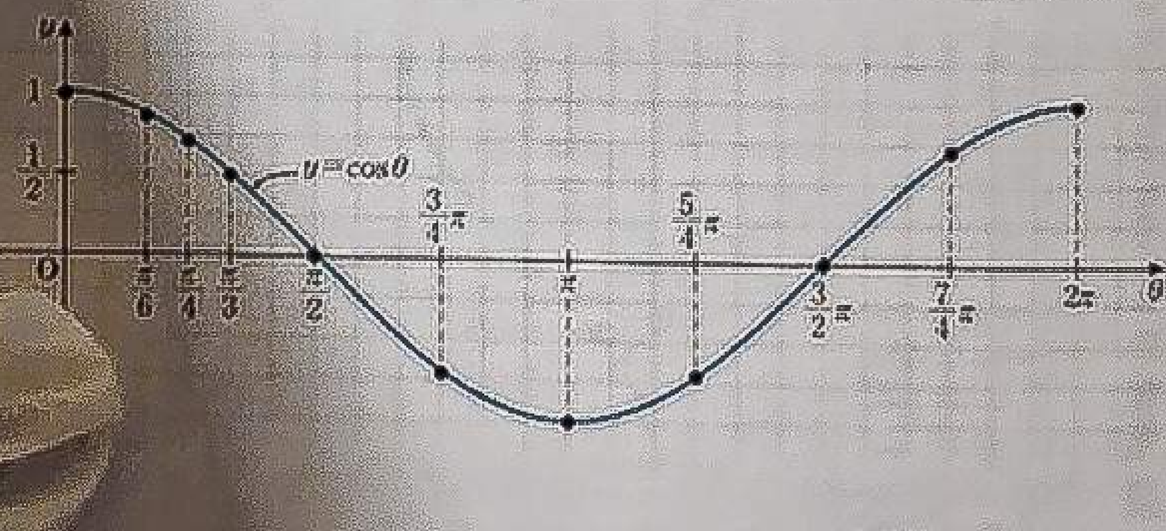




1. Draw the graph of $y = \cos\theta$ for $0 \leqq \theta \leqq 2\pi$.

[Sol]

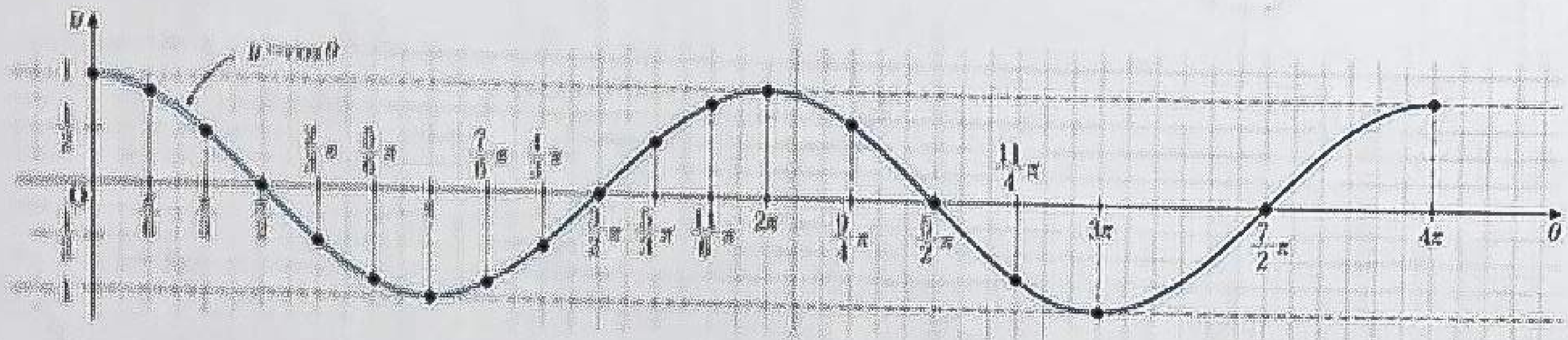
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
y	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1



M I 32b

graph of $y = \cos\theta$ for $0 \leqq \theta \leqq 4\pi$. Also, fill in the blanks below.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$	2π	$\frac{9}{4}\pi$	$\frac{5}{2}\pi$	$\frac{11}{4}\pi$	3π	$\frac{7}{2}\pi$	4π
y	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	0	1



The period of $y = \cos\theta$ is $\boxed{2\pi}$.

Also, based on the graph and the period above, when θ takes all real values, $\boxed{-1} \leqq \cos\theta \leqq \boxed{1}$ is true.

Graphs of Trigonometric Functions

Name

Date / /

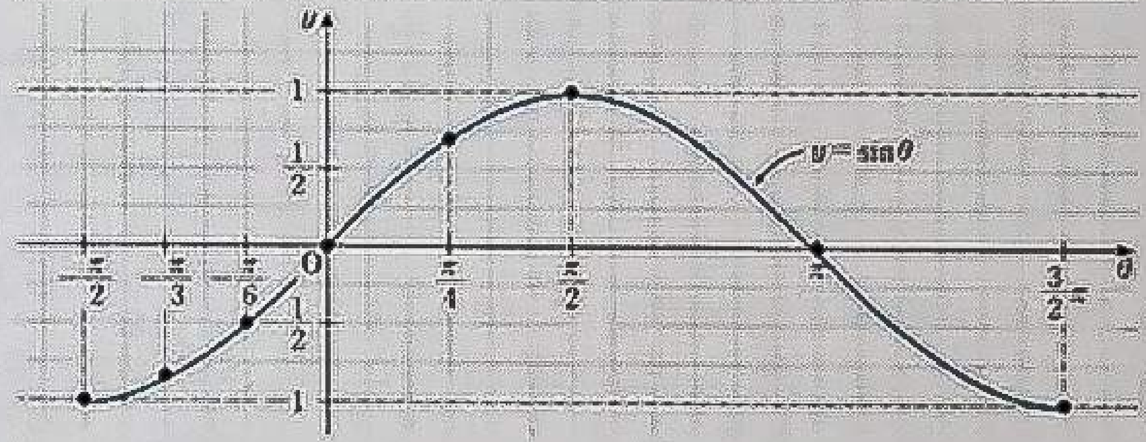
Time : to :

100%	~90%	~80%	~70%	69%~
0			1	2~

1. Draw the graph of $y=\sin\theta$ for $-\frac{\pi}{2} \leqq \theta \leqq \frac{3}{2}\pi$.

[Sol]

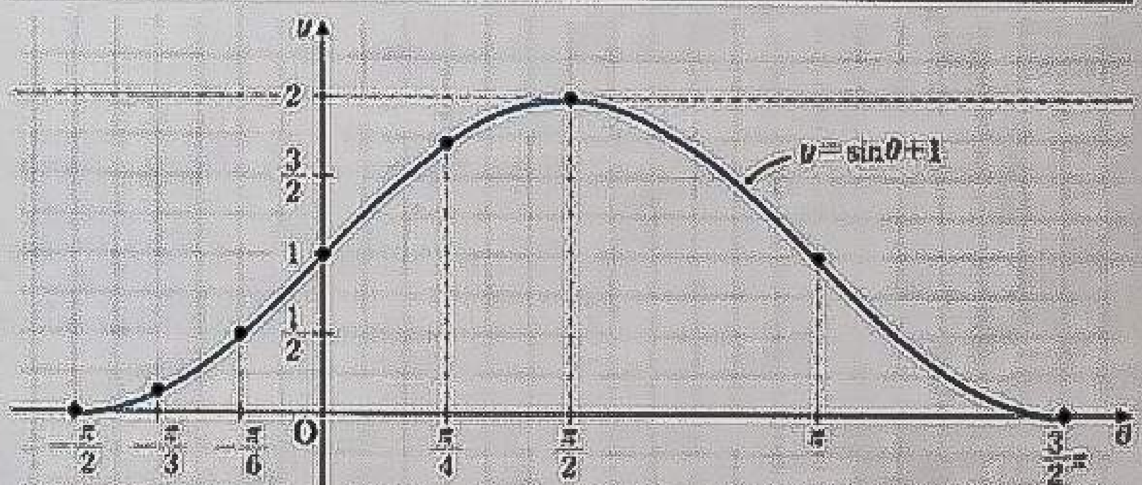
θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$
y	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{\sqrt{2}}{2}$	1	0	-1



2. Draw the graph of $y=\sin\theta+1$ for $-\frac{\pi}{2} \leqq \theta \leqq \frac{3}{2}\pi$.

[Sol]

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$
y	0	$-\frac{\sqrt{3}}{2}+1$	$\frac{1}{2}$	1	$\frac{\sqrt{2}}{2}+1$	2	1	0

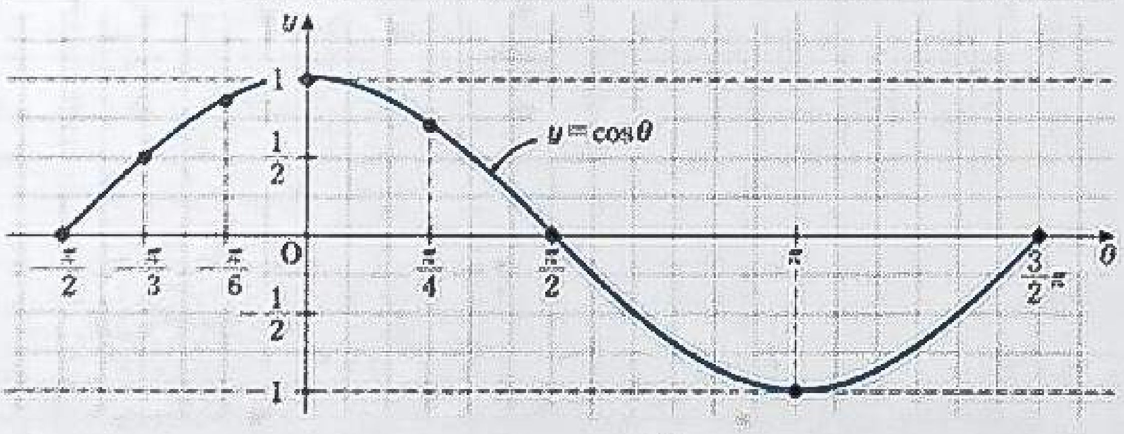


M133b

. Draw the graph of $y=\cos\theta$ for $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{3}{2}\pi$.

[Sol]

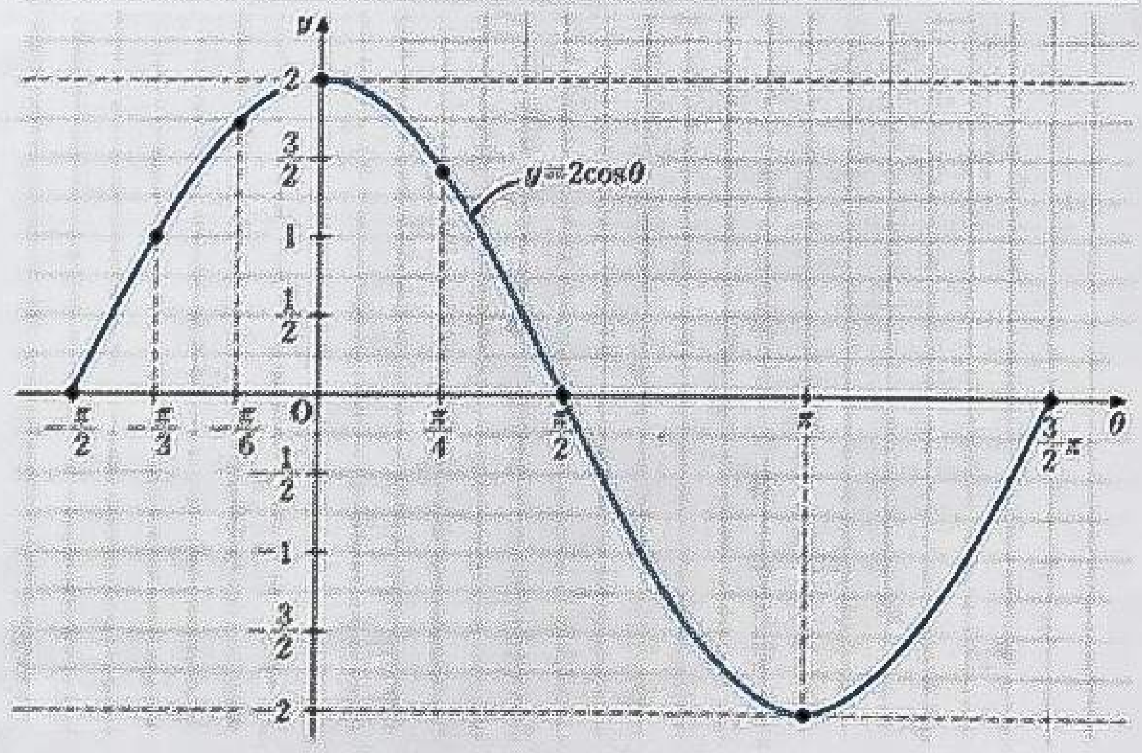
θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$
y	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	-1	0



. Draw the graph of $y=2\cos\theta$ for $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{3}{2}\pi$.

[Sol]

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$
y	0	1	$\sqrt{3}$	2	$\sqrt{2}$	0	-2	0



Graphs of Trigonometric Functions

Name

Date / /

Time : to :

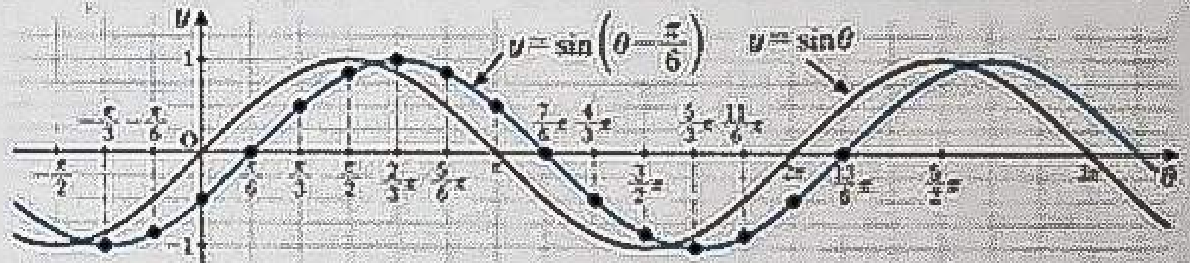
100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	1	2~

1. Draw the graph of $y=\sin\left(\theta-\frac{\pi}{6}\right)$.

Then, state its positional relationship with $y=\sin\theta$.

[Sol]

θ	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$	2π	$\frac{13}{6}\pi$
y	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0



The graph of $y=\sin\left(\theta-\frac{\pi}{6}\right)$ is a translation of the graph of $y=\sin\theta$,

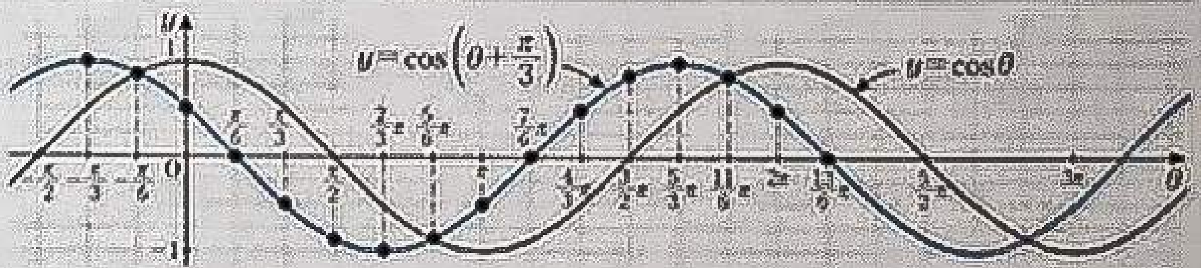
$\boxed{\frac{\pi}{6}}$ unit(s) along the θ-axis.

2. Draw the graph of $y=\cos\left(\theta+\frac{\pi}{3}\right)$.

Then, state its positional relationship with $y=\cos\theta$.

[Sol]

θ	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$	2π	$\frac{13}{6}\pi$
y	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0



The graph of $y=\cos\left(\theta+\frac{\pi}{3}\right)$ is a translation of the graph of $y=\cos\theta$,

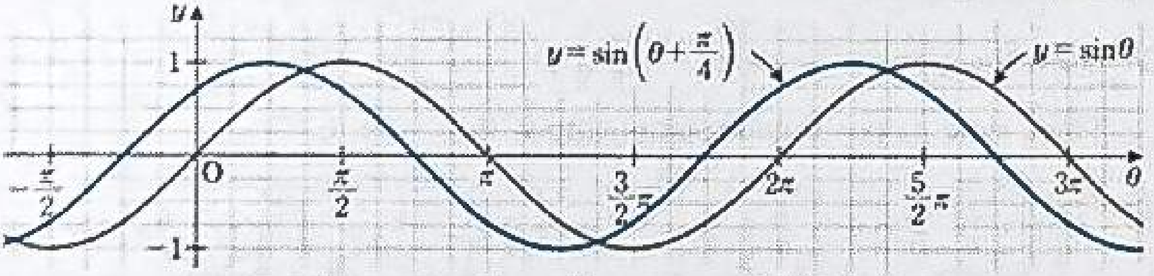
$\boxed{-\frac{\pi}{3}}$ unit(s) along the θ-axis. [Alternative Solution $\frac{5}{3}\pi$]

M134b

3. Draw the graph of $y=\sin\left(\theta+\frac{\pi}{4}\right)$.

Then, state its positional relationship with $y=\sin\theta$.

[Sol]



The graph of $y=\sin\left(\theta+\frac{\pi}{4}\right)$ is a translation of the graph of $y=\sin\theta$,

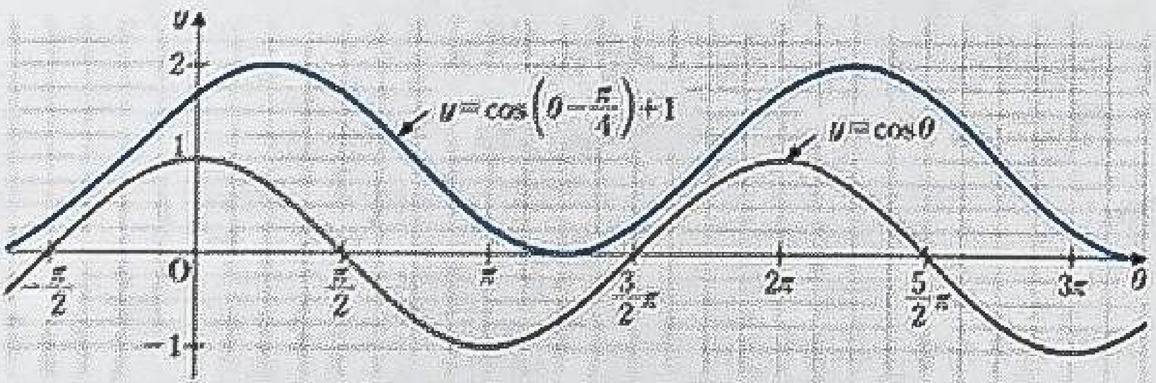
$\boxed{-\frac{\pi}{4}}$ unit(s) along the θ-axis.

[Alternative Solution $\frac{7}{4}\pi$]

. Draw the graph of $y=\cos\left(\theta-\frac{\pi}{4}\right)+1$.

Then, state its positional relationship with $y=\cos\theta$.

[Sol]



The graph of $y=\cos\left(\theta-\frac{\pi}{4}\right)+1$ is a translation of the graph of $y=\cos\theta$,

$\boxed{\frac{\pi}{4}}$ unit(s) along the θ-axis and $\boxed{1}$ unit(s) along the y-axis.

Graphs of Trigonometric Functions

Name

Date / /

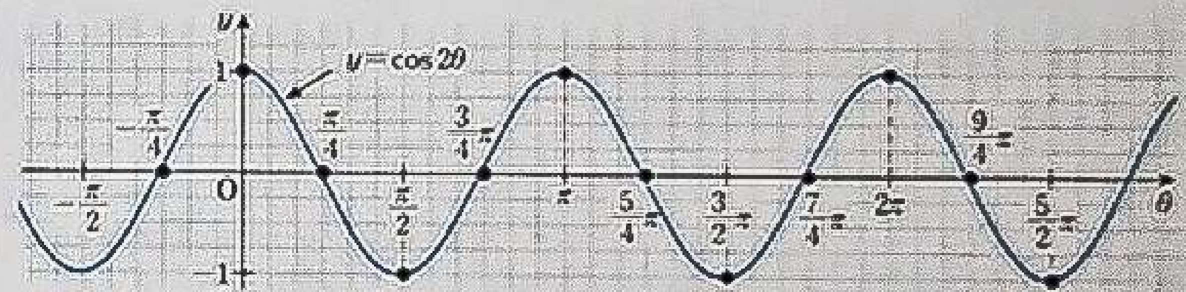
Time : to :

100%	~90%	~80%	~70%	69%~
mistakes 0	—	—	1	2~

1. Draw the graph of $y=\cos 2\theta$. Then, state its period.

[Sol]

θ	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π	$\frac{9}{4}\pi$	$\frac{5}{2}\pi$
y	0	1	0	-1	0	1	0	-1	0	1	0	-1

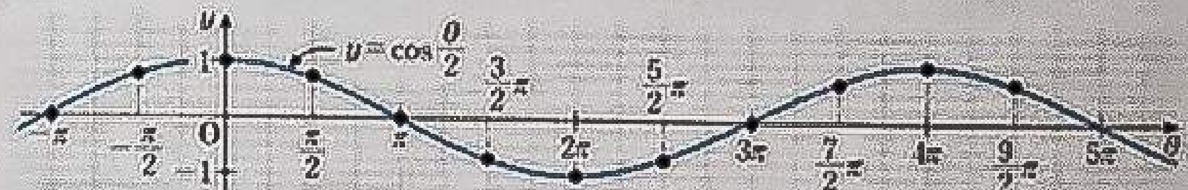


The period of $y=\cos 2\theta$ is $\boxed{\pi}$.

2. Draw the graph of $y=\cos\frac{\theta}{2}$. Then, state its period.

[Sol]

θ	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π	$\frac{5}{2}\pi$	3π	$\frac{7}{2}\pi$	4π	$\frac{9}{2}\pi$
y	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$

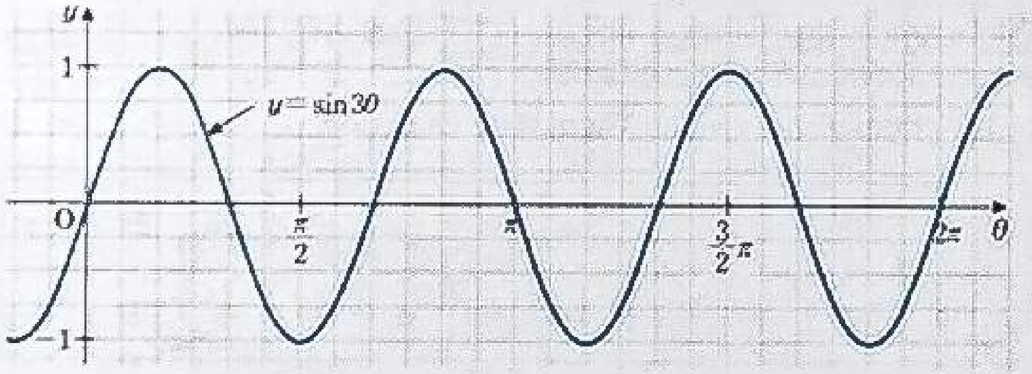


The period of $y=\cos\frac{\theta}{2}$ is $\boxed{4\pi}$.

M135b

3. Draw the graph of $y = \sin 3\theta$. Then, state its period.

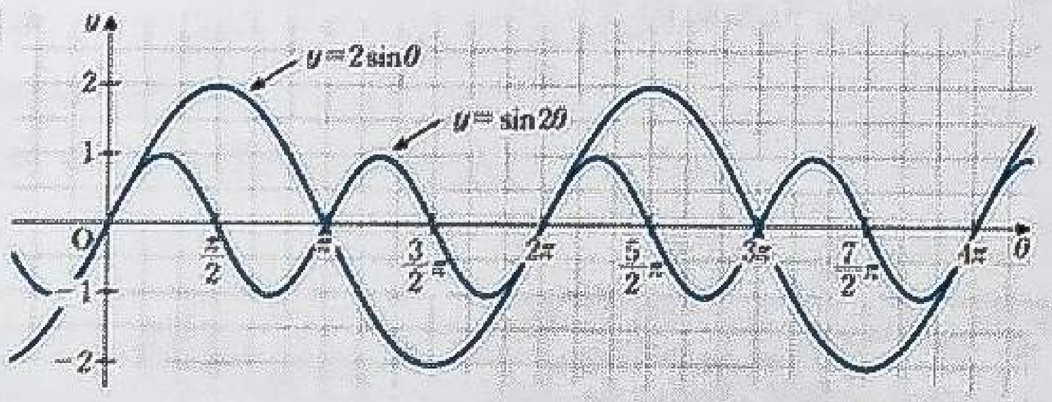
[Sol]



The period of $y = \sin 3\theta$ is $\boxed{\frac{2}{3}\pi}$.

4. Draw the graphs of $y = \sin 2\theta$ and $y = 2\sin\theta$. Then, state their periods.

[Sol]



The period of $y = \sin 2\theta$ is $\boxed{\pi}$.

The period of $y = 2\sin\theta$ is $\boxed{2\pi}$.

Graphs of Trigonometric Functions

Name

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Time : to :

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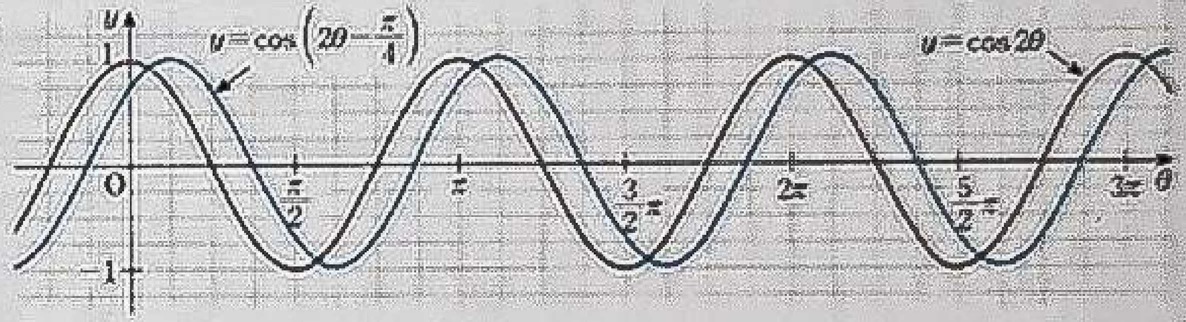
mistakes 0	—	—	1	2~

1. Draw the graph of $y=\cos\left(2\theta-\frac{\pi}{4}\right)$ and state its period.

Then, state its positional relationship with $y=\cos 2\theta$.

[Sol]



The period of $y=\cos\left(2\theta-\frac{\pi}{4}\right)$ is $\boxed{\pi}$.

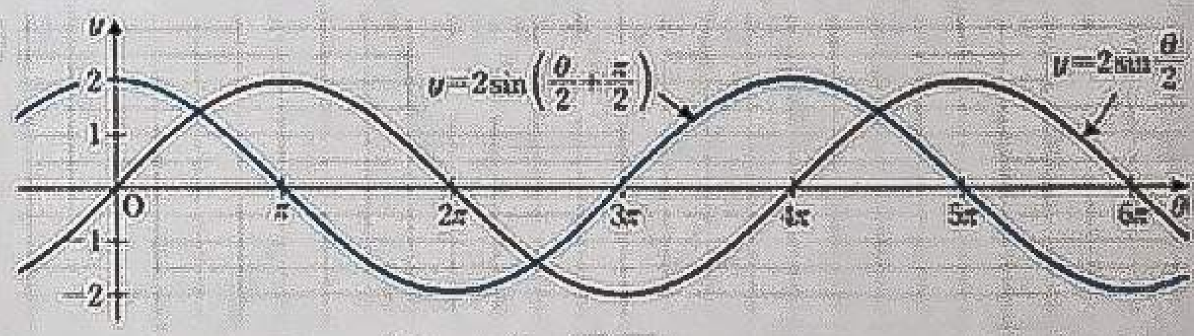
The graph of $y=\cos\left(2\theta-\frac{\pi}{4}\right)$ is a translation of the graph of $y=\cos 2\theta$,

$\boxed{\frac{\pi}{8}}$ unit(s) along the θ-axis. ⟵ $y=\cos\left(2\theta-\frac{\pi}{4}\right)=\cos 2\left(\theta-\frac{\pi}{8}\right)$

2. Draw the graph of $y=2\sin\left(\frac{\theta}{2}+\frac{\pi}{2}\right)$ and state its period.

Then, state its positional relationship with $y=2\sin\frac{\theta}{2}$.

[Sol]



The period of $y=2\sin\left(\frac{\theta}{2}+\frac{\pi}{2}\right)$ is $\boxed{4\pi}$.

The graph of $y=2\sin\left(\frac{\theta}{2}+\frac{\pi}{2}\right)$ is a translation of the graph of $y=2\sin\frac{\theta}{2}$,

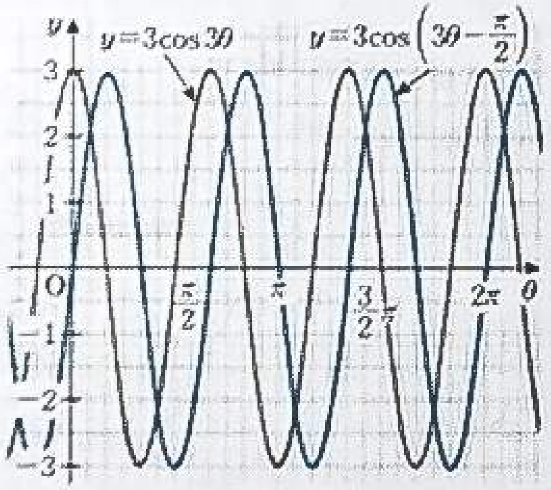
$\boxed{-\pi}$ unit(s) along the θ-axis.

[Alternative Solution 3π]

3. Draw the graph of $y=3\cos\left(3\theta-\frac{\pi}{2}\right)$ and state its period.

Then, state its positional relationship with $y=3\cos 3\theta$.

[Sol]

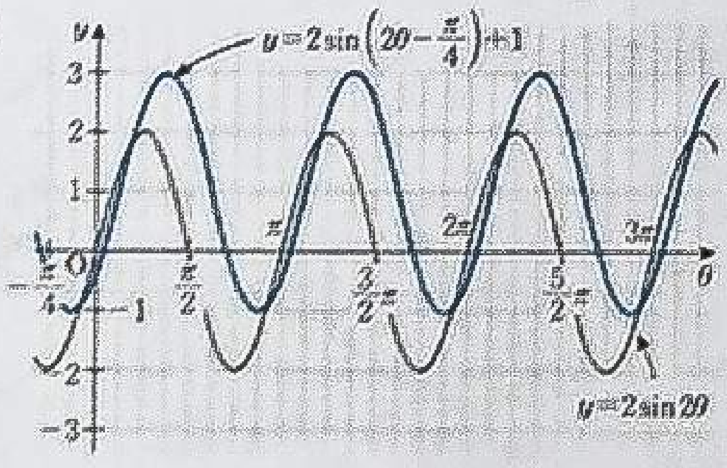


The period of $y=3\cos\left(3\theta-\frac{\pi}{2}\right)$ is $\boxed{\frac{2}{3}\pi}$. The graph of $y=3\cos\left(3\theta-\frac{\pi}{2}\right)$ is a translation of the graph of $y=3\cos 3\theta$, $\boxed{\frac{\pi}{6}}$ unit(s) along the θ-axis.

4. Draw the graph of $y=2\sin\left(2\theta-\frac{\pi}{4}\right)+1$ and state its period.

Then, state its positional relationship with $y=2\sin 2\theta$.

[Sol]



The period of $y=2\sin\left(2\theta-\frac{\pi}{4}\right)+1$ is $\boxed{\pi}$.

The graph of $y=2\sin\left(2\theta-\frac{\pi}{4}\right)+1$ is a translation of the graph of $y=2\sin 2\theta$, $\boxed{\frac{\pi}{8}}$ unit(s) along the θ-axis and $\boxed{1}$ unit(s) along the y-axis.

Graphs of Trigonometric Functions

Name

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Time : to :

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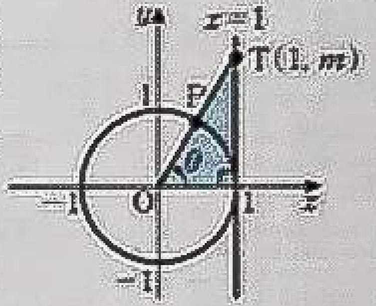
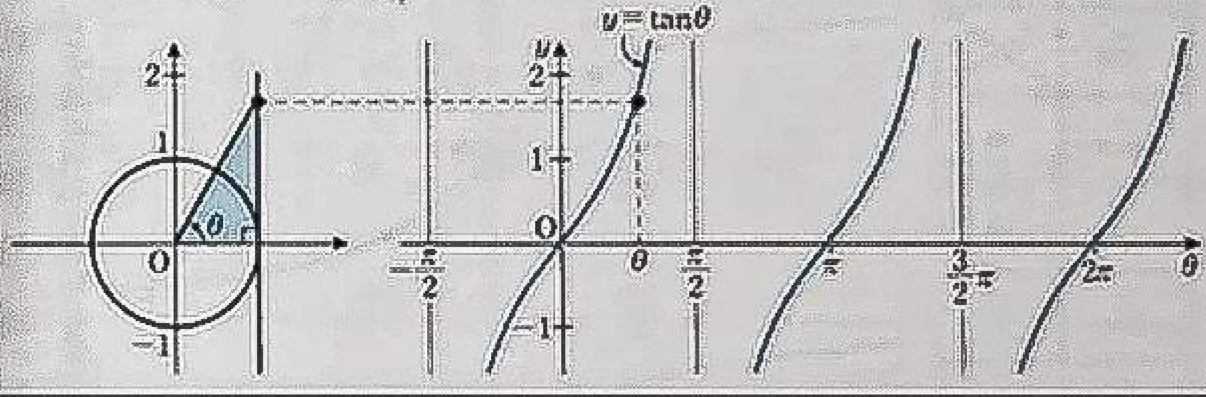
Graph of $y = \tan\theta$

Let the point of intersection of the terminal side of θ and the unit circle be P, and let the point of intersection of line OP and line $x=1$ be $T(1, m)$.
Then, $\tan\theta = m$ is true.
Using this, it is possible to draw the graph of $y = \tan\theta$.

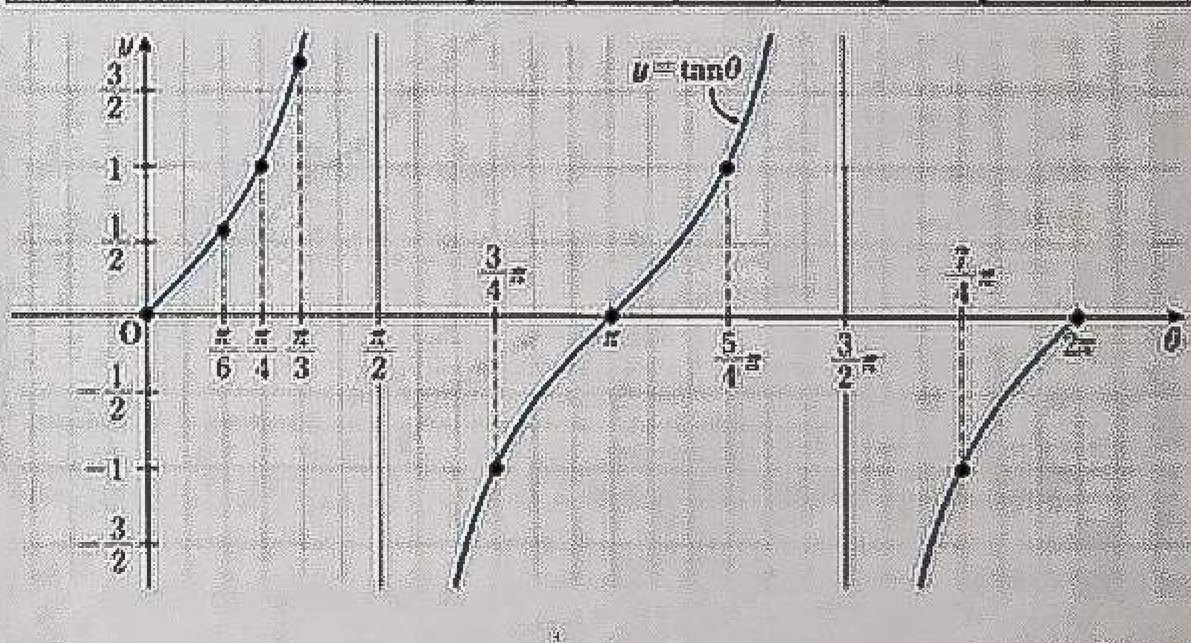




1. Draw the graph of $y = \tan\theta$ for $0 \leqq \theta \leqq 2\pi$.

[Sol]

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
y	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\times$	-1	0	1	$\times$	-1	0

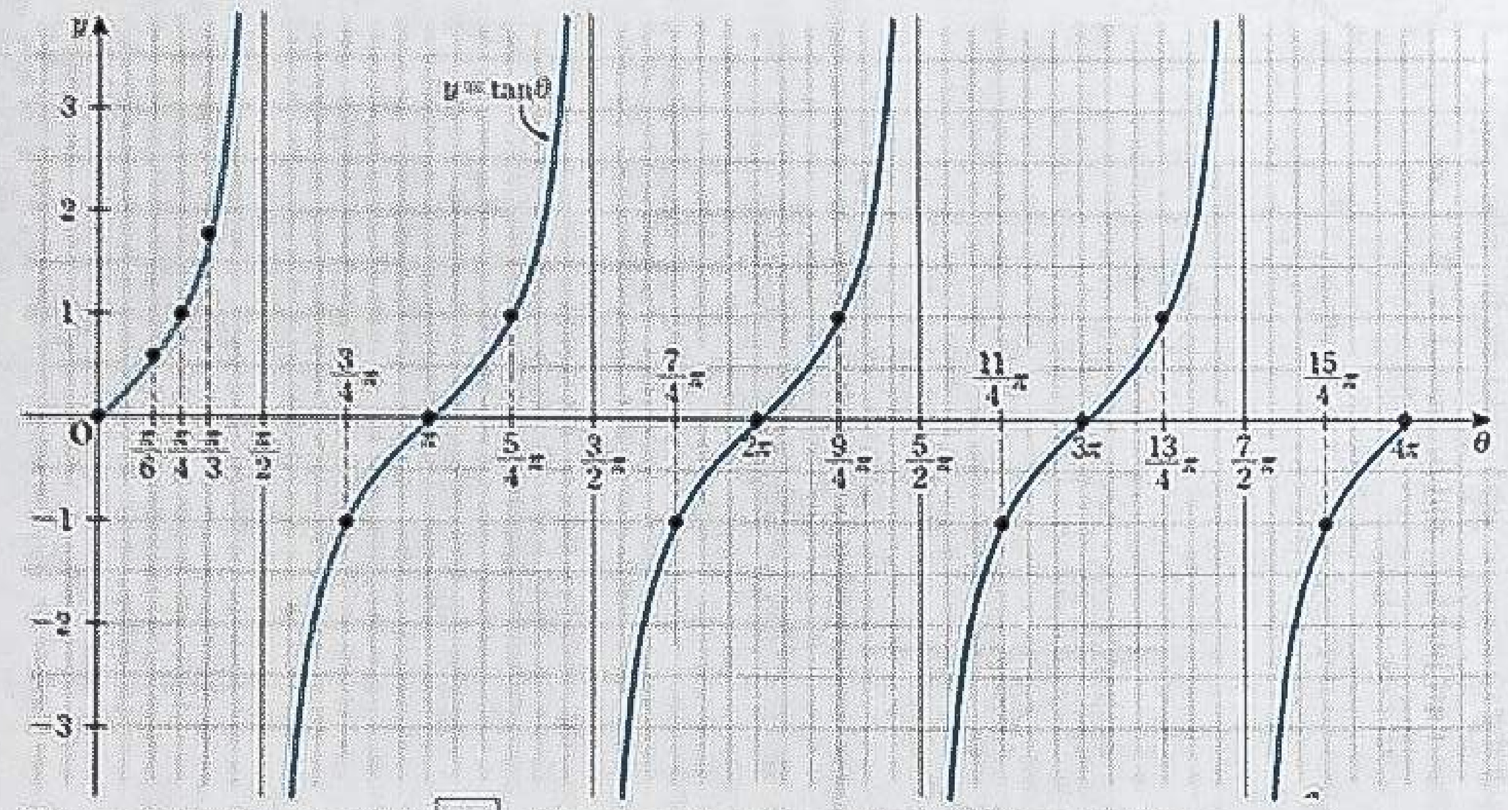


M137b

2. Draw the graph of $y = \tan\theta$ for $0 \leqq \theta \leqq 4\pi$. Then, state its period.

[Sol]

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π	$\frac{9}{4}\pi$	$\frac{5}{2}\pi$	$\frac{11}{4}\pi$	3π	$\frac{13}{4}\pi$	$\frac{7}{2}\pi$	$\frac{15}{4}\pi$	4π
y	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	╳	-1	0	1	╳	-1	0	1	╳	-1	0	1	╳	-1	0



The period of $y = \tan\theta$ is $\boxed{\pi}$. Also, based on the period of the graph above,
when θ takes all real values except $\theta = \frac{\pi}{2} + n\pi$ (where n is an integer), the range of $\tan\theta$ is equal to all real values.

Graphs of Trigonometric Functions

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Name

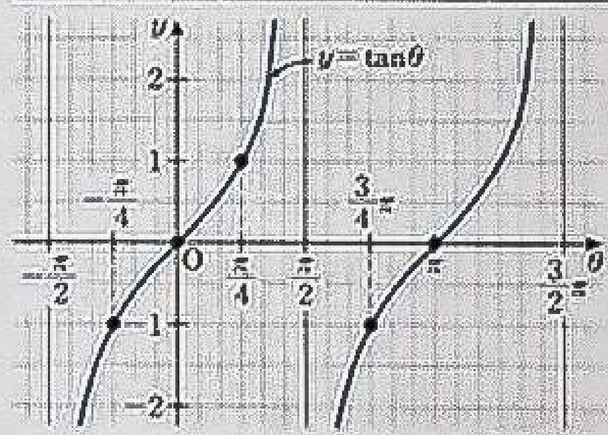
Date / /

Time : to :

1. Draw the graph of $y = \tan\theta$ for $-\frac{\pi}{2} \leqq \theta \leqq \frac{3}{2}\pi$.

[Sol]

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{3}{2}\pi$
y	╳	-1	0	1	╳	-1	0	╳

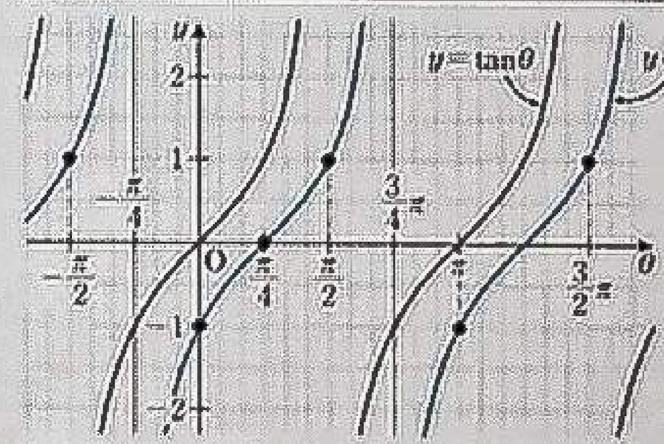


2. Draw the graph of $y = \tan\left(\theta - \frac{\pi}{4}\right)$.

Then, state its positional relationship with $y = \tan\theta$.

[Sol]

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{3}{2}\pi$
y	1	╳	-1	0	1	╳	-1	1



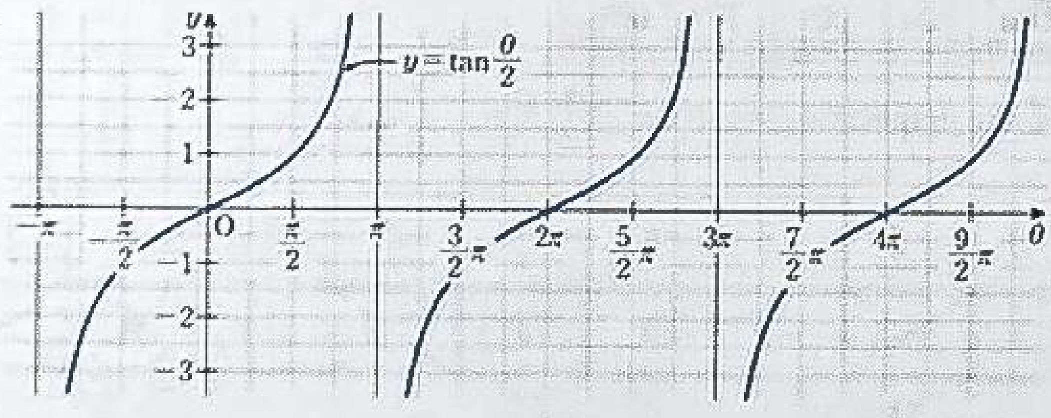
The graph of $y = \tan\left(\theta - \frac{\pi}{4}\right)$ is a translation of the graph of $y = \tan\theta$.

$\boxed{\frac{\pi}{4}}$ unit(s) along the θ-axis.

M138b

3. Draw the graph of $y = \tan\frac{\theta}{2}$. Then, state its period.

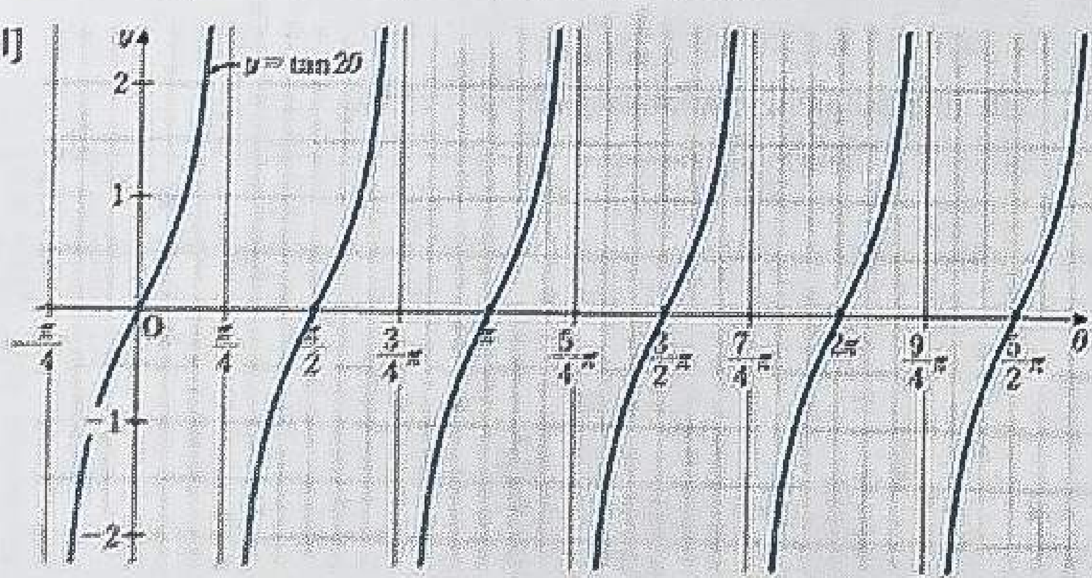
[Sol]



The period of $y = \tan\frac{\theta}{2}$ is $\boxed{2\pi}$.

4. Draw the graph of $y = \tan 2\theta$. Then, state its period.

[Sol]



The period of $y = \tan 2\theta$ is $\boxed{\frac{\pi}{2}}$.

Graphs of Trigonometric Functions

Name

Date / /

Time : to :

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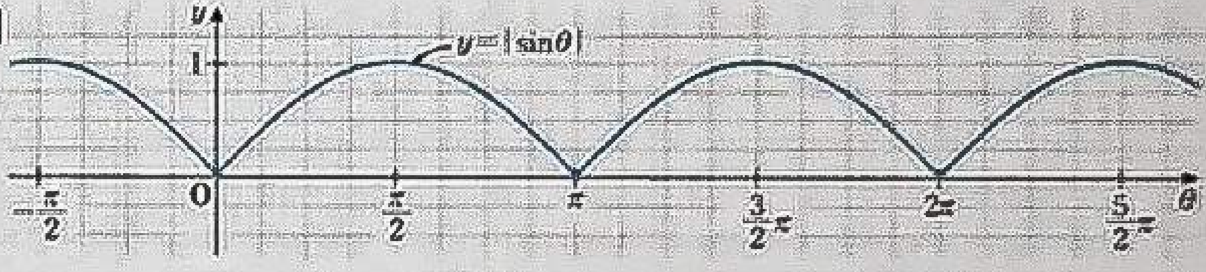
(mistakes) 0	—	—	1	2~

1. Draw the graph of $y=|\sin\theta|$. Then, state its period.

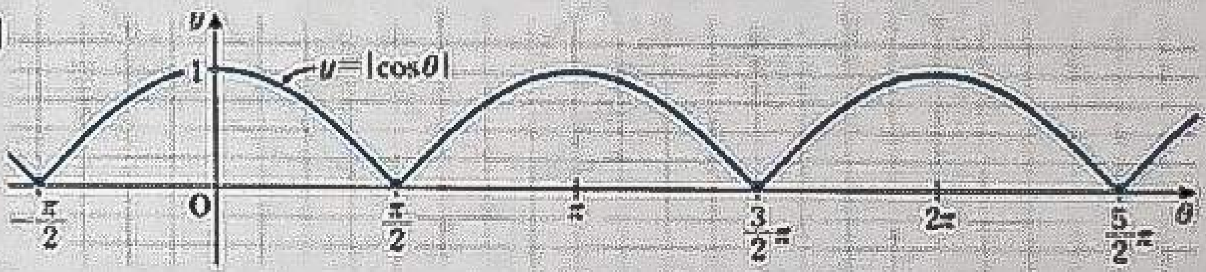
[Sol]



The period of $y=|\sin\theta|$ is $\boxed{\pi}$.

2. Draw the graph of $y=|\cos\theta|$. Then, state its period.

[Sol]

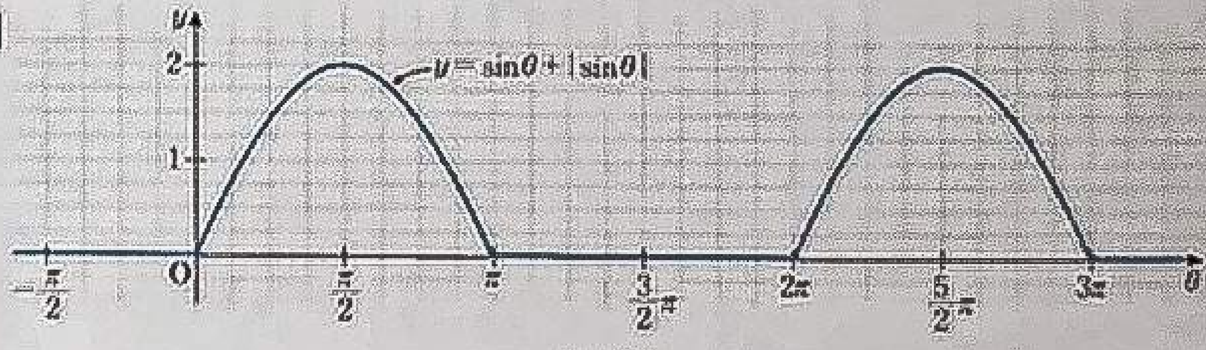


The period of $y=|\cos\theta|$ is $\boxed{\pi}$.

When $\sin\theta>0$, $y=2\sin\theta$
When $\sin\theta<0$, $y=0$

3. Draw the graph of $y=\sin\theta+|\sin\theta|$. Then, state its period.

[Sol]



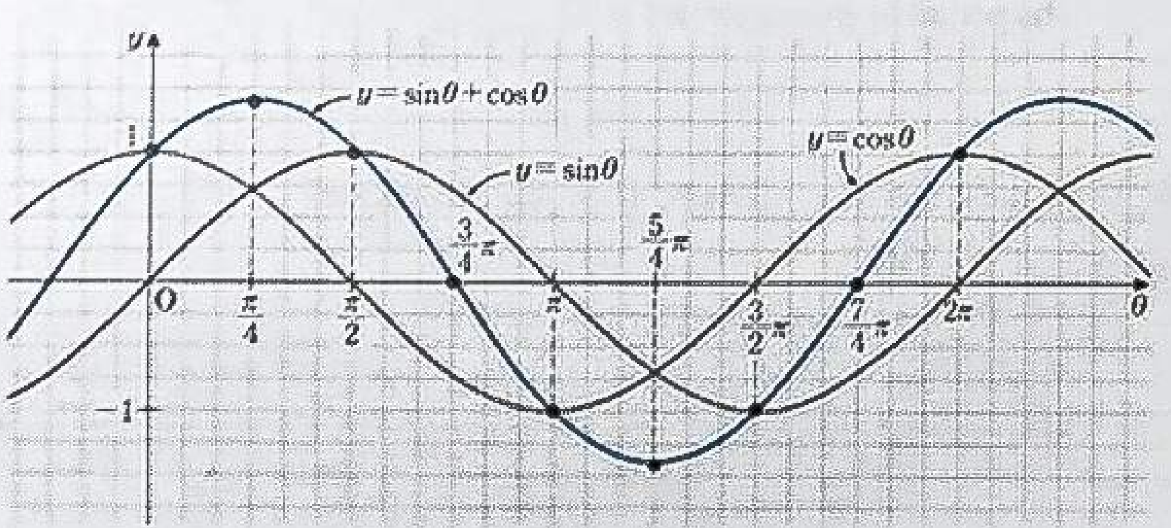
The period of $y=\sin\theta+|\sin\theta|$ is $\boxed{2\pi}$.

M139b

4. Draw the graph of $y = \sin\theta + \cos\theta$.

[Sol]

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
$\sin\theta$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1
$\sin\theta+\cos\theta$	1	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	-1	0	1



Graphs of Trigonometric Functions

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Name

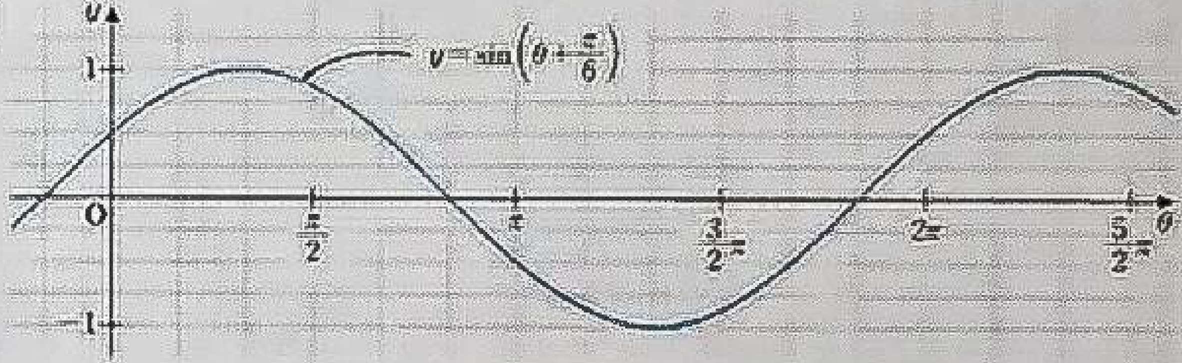
Date / /

Time : to :

1. Draw the graph of $y=\sin\left(\theta+\frac{\pi}{6}\right)$.

   Then, state its positional relationship with $y=\sin\theta$. ➡ M I 34

[Sol]



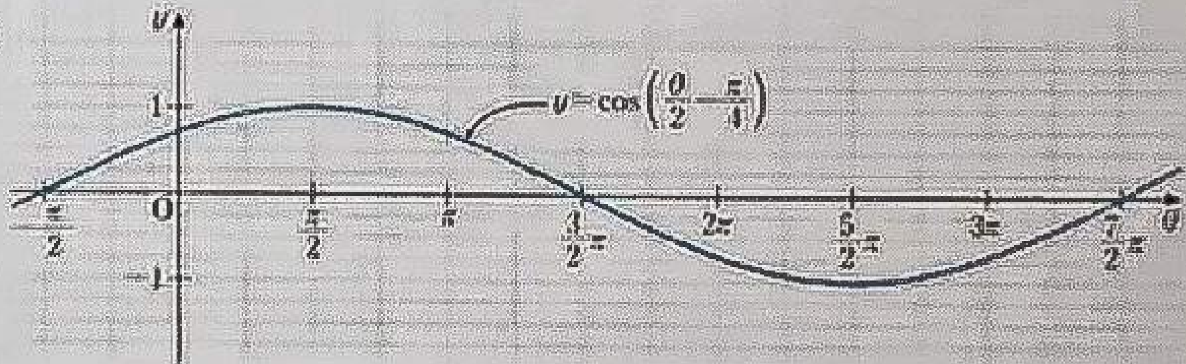
The graph of $y=\sin\left(\theta+\frac{\pi}{6}\right)$ is a translation of the graph of $y=\sin\theta$,

$\boxed{-\frac{\pi}{6}}$ unit(s) along the θ-axis. $\left[\begin{array}{ll}\text{Alternative Solution} & \frac{11}{6}\pi\end{array}\right]$

2. Draw the graph of $y=\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$ and state its period.

   Then, state its positional relationship with $y=\cos\frac{\theta}{2}$. ➡ M I 36

[Sol]



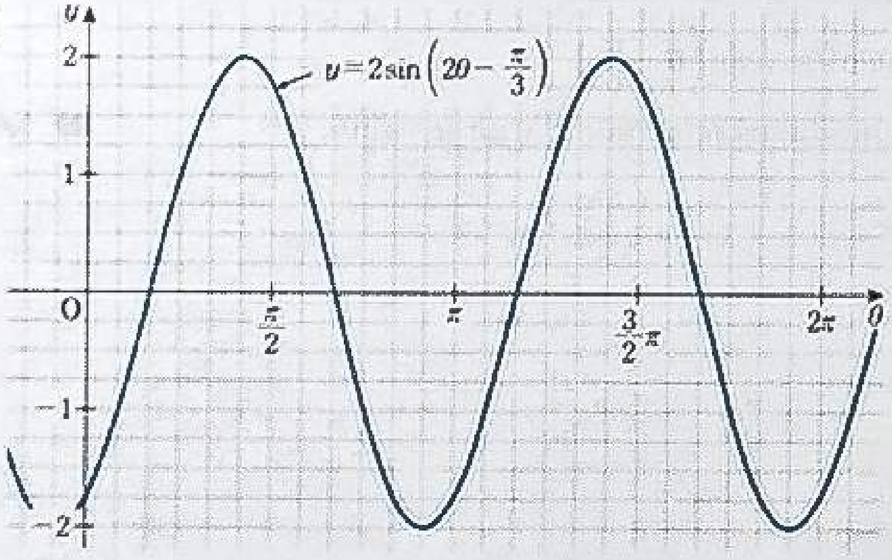
The period of $y=\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$ is $\boxed{4\pi}$.

The graph of $y=\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$ is a translation of the graph of $y=\cos\frac{\theta}{2}$.

$\boxed{\frac{\pi}{2}}$ unit(s) along the θ-axis.

3. Draw the graph of $y=2\sin\left(2\theta-\frac{\pi}{3}\right)$ and state its period.
Then, state its positional relationship with $y=2\sin 2\theta$. ➡ M136

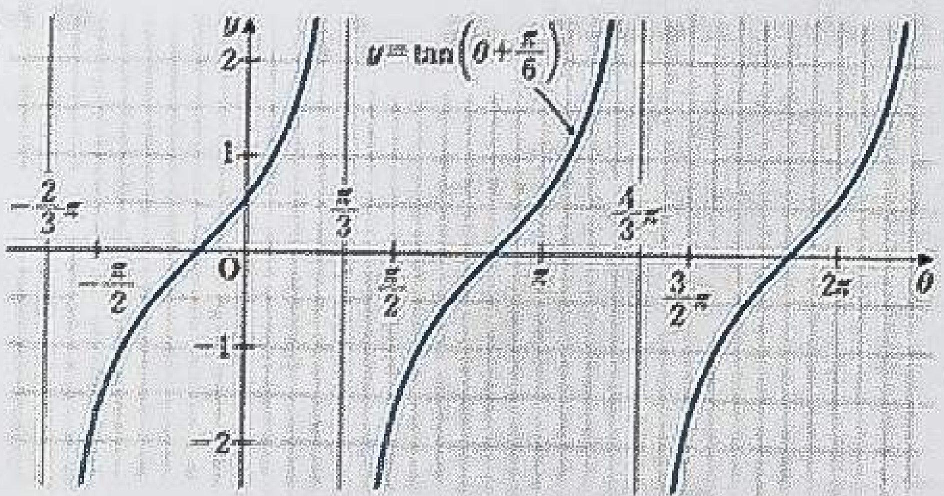
[Sol]



The period of $y=2\sin\left(2\theta-\frac{\pi}{3}\right)$ is $\boxed{\pi}$. The graph of $y=2\sin\left(2\theta-\frac{\pi}{3}\right)$ is a translation of the graph of $y=2\sin 2\theta$, $\boxed{\frac{\pi}{6}}$ unit(s) along the θ-axis.

4. Draw the graph of $y=\tan\left(\theta+\frac{\pi}{6}\right)$.
Then, state its positional relationship with $y=\tan\theta$. ➡ M138

[Sol]



The graph of $y=\tan\left(\theta+\frac{\pi}{6}\right)$ is a translation of the graph of $y=\tan\theta$, $\boxed{-\frac{\pi}{6}}$ unit(s) along the θ-axis. [Alternative Solution $\frac{5}{6}\pi$]

Trigonometric Inequalities

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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Given $0 \leqq \theta < 2\pi$, solve the following inequalities.

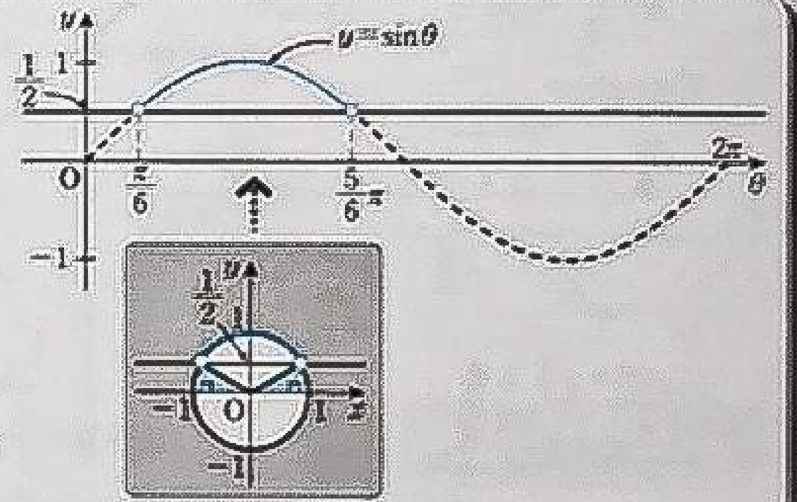
Ex. $\sin\theta > \frac{1}{2}$

[Sol] When $\sin\theta = \frac{1}{2}$,

$\theta = \frac{\pi}{6},\ \frac{5}{6}\pi$

Therefore,

$\frac{\pi}{6} < \theta < \frac{5}{6}\pi$



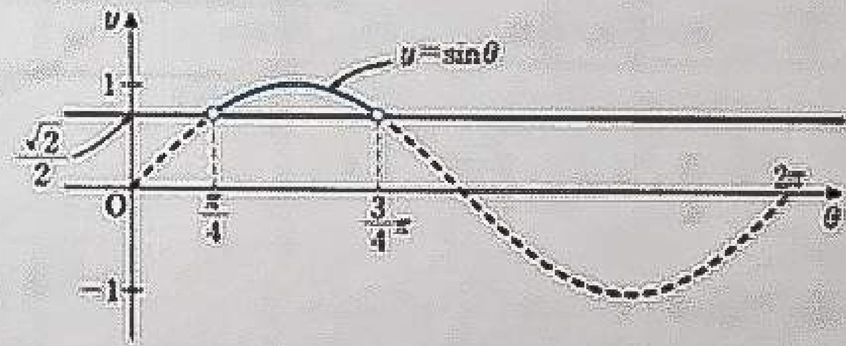
(1) $\sin\theta > \frac{\sqrt{2}}{2}$

[Sol] When $\sin\theta = \frac{\sqrt{2}}{2}$,

$\theta = \frac{\pi}{4},\ \frac{3}{4}\pi$

Therefore,

$\frac{\pi}{4} < \theta < \frac{3}{4}\pi$



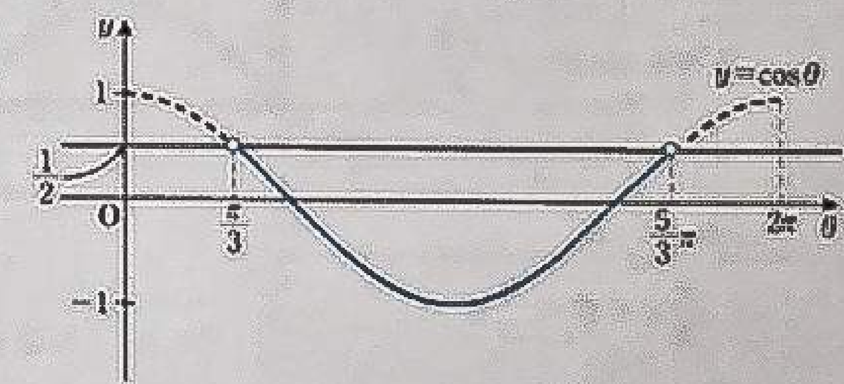
(2) $\cos\theta < \frac{1}{2}$

[Sol] When $\cos\theta = \frac{1}{2}$,

$\theta = \frac{\pi}{3},\ \frac{5}{3}\pi$

Therefore,

$\frac{\pi}{3} < \theta < \frac{5}{3}\pi$



M141b

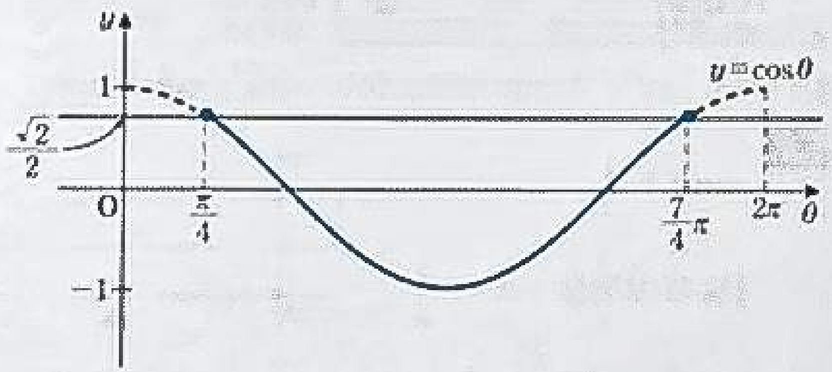
(3) $\cos\theta \leqq \frac{\sqrt{2}}{2}$

[Sol] When $\cos\theta = \frac{\sqrt{2}}{2}$,

$\theta = \frac{\pi}{4}, \frac{7}{4}\pi$

Therefore,

$\frac{\pi}{4} \leqq \theta \leqq \frac{7}{4}\pi$



(4) $2\cos\theta + \sqrt{2} \leqq 0$

[Sol] Rearranging,

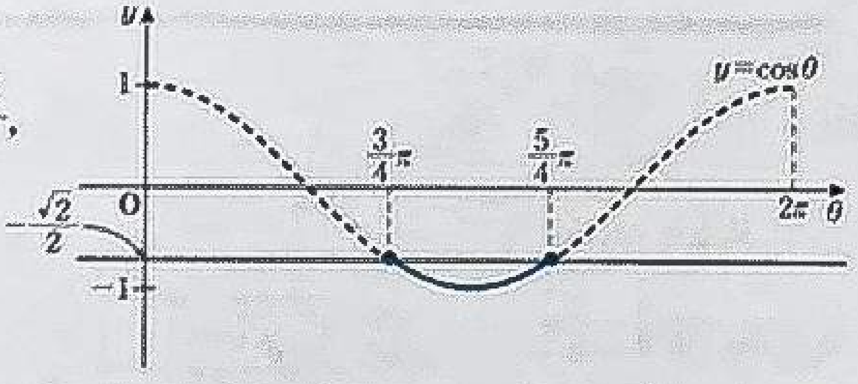
$\cos\theta \leqq -\frac{\sqrt{2}}{2}$

When $\cos\theta = -\frac{\sqrt{2}}{2}$,

$\theta = \frac{3}{4}\pi, \frac{5}{4}\pi$

Therefore,

$\frac{3}{4}\pi \leqq \theta \leqq \frac{5}{4}\pi$



(5) $2\sin\theta + \sqrt{3} < 0$

[Sol] Rearranging,

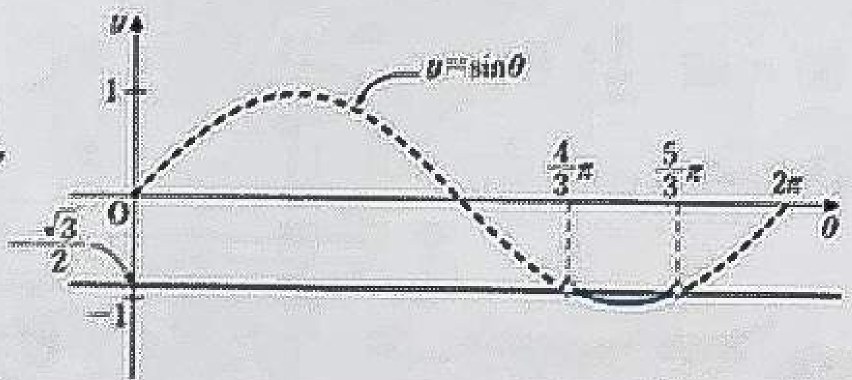
$\sin\theta < -\frac{\sqrt{3}}{2}$

When $\sin\theta = -\frac{\sqrt{3}}{2}$,

$\theta = \frac{4}{3}\pi, \frac{5}{3}\pi$

Therefore,

$\frac{4}{3}\pi < \theta < \frac{5}{3}\pi$



Trigonometric Inequalities

Name

Date / /

Time : to :

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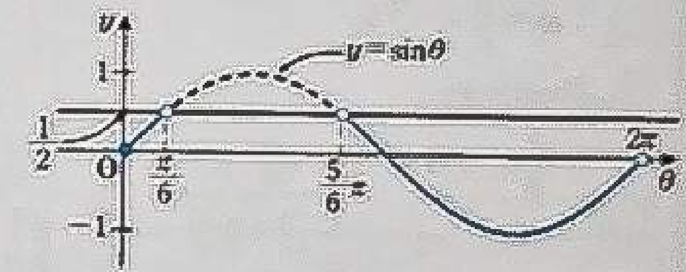
Given $0 \leqslant \theta < 2\pi$, solve the following inequalities.

(1) $\sin\theta < \frac{1}{2}$

[Sol] When $\sin\theta = \frac{1}{2}$,



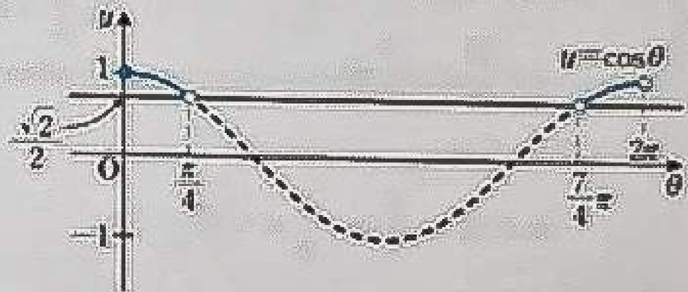
$\theta = \boxed{\frac{\pi}{6}}, \boxed{\frac{5}{6}\pi}$

Therefore,

$0 \leqslant \theta < \boxed{\frac{\pi}{6}}, \boxed{\frac{5}{6}\pi} < \theta < 2\pi$ ← Since $0 \leqslant \theta < 2\pi$, $\theta = 0$ is included, but $\theta = 2\pi$ is not included.

(2) $\cos\theta > \frac{\sqrt{2}}{2}$

[Sol] When $\cos\theta = \frac{\sqrt{2}}{2}$,



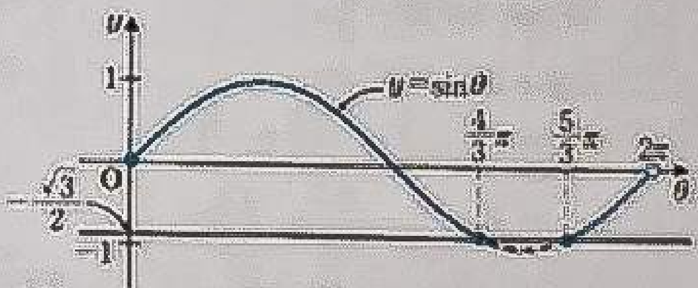
$\theta = \frac{\pi}{4}, \frac{7}{4}\pi$

Therefore,

$0 \leqslant \theta < \frac{\pi}{4}, \frac{7}{4}\pi < \theta < 2\pi$

(3) $\sin\theta \geqslant -\frac{\sqrt{3}}{2}$

[Sol] When $\sin\theta = -\frac{\sqrt{3}}{2}$,



$\theta = \frac{4}{3}\pi, \frac{5}{3}\pi$

Therefore,

$0 \leqslant \theta \leqslant \frac{4}{3}\pi, \frac{5}{3}\pi \leqslant \theta < 2\pi$

M142b

(4) $2\cos\theta+1<0$

[Sol] Rearranging,

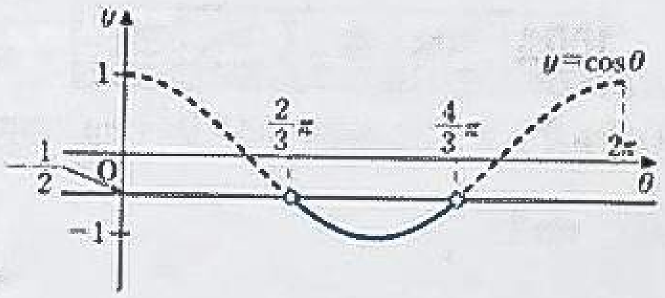
$$\cos\theta<-\frac{1}{2}$$

When $\cos\theta=-\frac{1}{2}$,

$$\theta=\frac{2}{3}\pi,\ \frac{4}{3}\pi$$

Therefore,

$$\frac{2}{3}\pi<\theta<\frac{4}{3}\pi$$



(5) $2\sin\theta+\sqrt{2}>0$

[Sol] Rearranging,

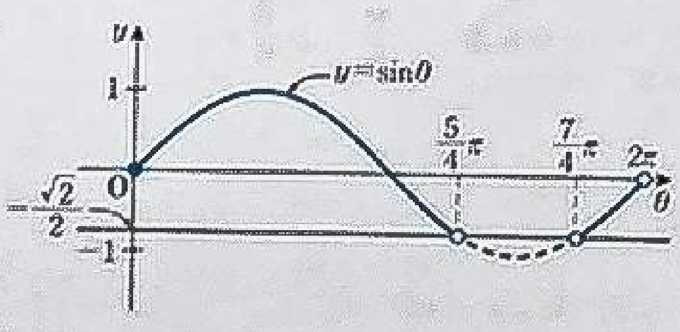
$$\sin\theta>-\frac{\sqrt{2}}{2}$$

When $\sin\theta=-\frac{\sqrt{2}}{2}$,

$$\theta=\frac{5}{4}\pi,\ \frac{7}{4}\pi$$

Therefore,

$$0\leqq\theta<\frac{5}{4}\pi,\ \frac{7}{4}\pi<\theta<2\pi$$



6) $2\sin 2\theta-1\leqq 0$

Sol] Rearranging,

$$\sin 2\theta\leqq\frac{1}{2}$$

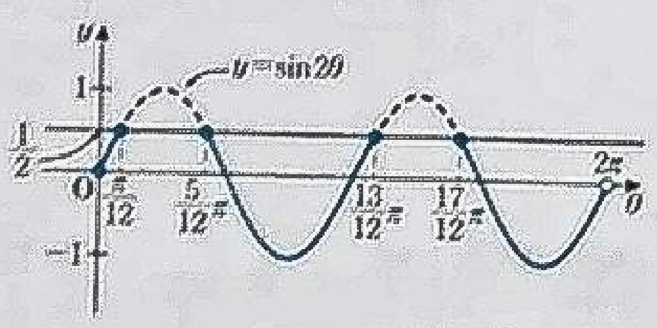
When $\sin 2\theta=\frac{1}{2}$,

since $0\leqq 2\theta<4\pi$,

$$2\theta=\frac{\pi}{6},\ \frac{5}{6}\pi,\ \frac{13}{6}\pi,\ \frac{17}{6}\pi$$

$$\therefore\ \theta=\frac{\pi}{12},\ \frac{5}{12}\pi,\ \frac{13}{12}\pi,\ \frac{17}{12}\pi$$

$$\therefore\ 0\leqq\theta\leqq\frac{\pi}{12},\ \frac{5}{12}\pi\leqq\theta\leqq\frac{13}{12}\pi,\ \frac{17}{12}\pi\leqq\theta<2\pi$$



Trigonometric Inequalities

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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Given $0 \leqq \theta < 2\pi$, solve the following inequalities.

Ex.

$-\frac{1}{2} < \cos\theta < \frac{1}{2}$

[Sol] When $\cos\theta = -\frac{1}{2}$,

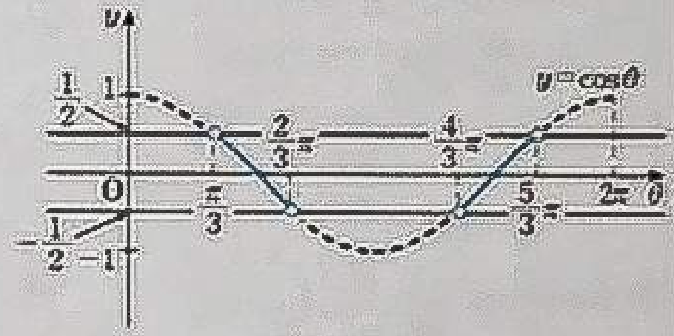
$\theta = \frac{2}{3}\pi,\ \frac{4}{3}\pi$

When $\cos\theta = \frac{1}{2}$,

$\theta = \frac{\pi}{3},\ \frac{5}{3}\pi$

Therefore,

$\frac{\pi}{3} < \theta < \frac{2}{3}\pi,\ \frac{4}{3}\pi < \theta < \frac{5}{3}\pi$



(1) $-\frac{\sqrt{3}}{2} < \cos\theta < \frac{\sqrt{2}}{2}$

[Sol] When $\cos\theta = -\frac{\sqrt{3}}{2}$,

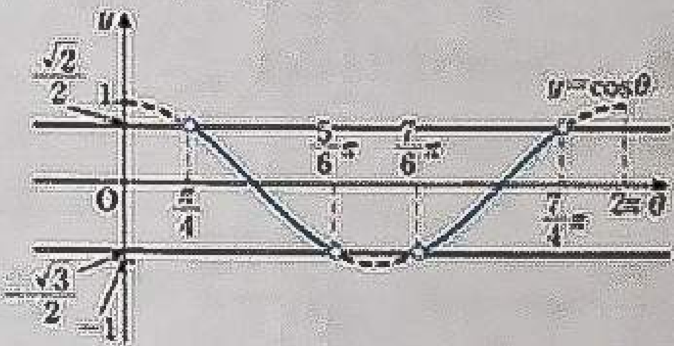
$\theta = \frac{5}{6}\pi,\ \frac{7}{6}\pi$

When $\cos\theta = \frac{\sqrt{2}}{2}$,

$\theta = \frac{\pi}{4},\ \frac{7}{4}\pi$

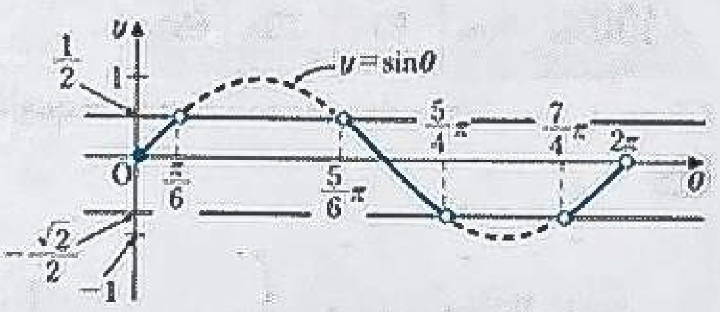
Therefore,

$\frac{\pi}{4} < \theta < \frac{5}{6}\pi,\ \frac{7}{6}\pi < \theta < \frac{7}{4}\pi$



M143b

(2) $-\frac{\sqrt{2}}{2} < \sin\theta < \frac{1}{2}$



[Sol] When $\sin\theta = -\frac{\sqrt{2}}{2}$,

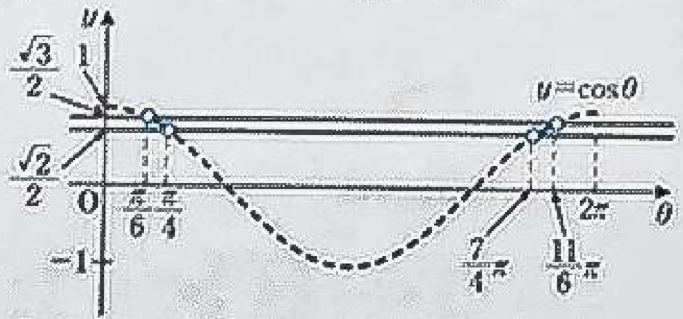
$\theta = \frac{5}{4}\pi,\ \frac{7}{4}\pi$

When $\sin\theta = \frac{1}{2}$,

$\theta = \frac{\pi}{6},\ \frac{5}{6}\pi$

$\therefore\ 0 \leqq \theta < \frac{\pi}{6},\ \frac{5}{6}\pi < \theta < \frac{5}{4}\pi,\ \frac{7}{4}\pi < \theta < 2\pi$

(3) $\sqrt{2} < 2\cos\theta < \sqrt{3}$



[Sol] Rearranging,

$\frac{\sqrt{2}}{2} < \cos\theta < \frac{\sqrt{3}}{2}$

When $\cos\theta = \frac{\sqrt{2}}{2}$,

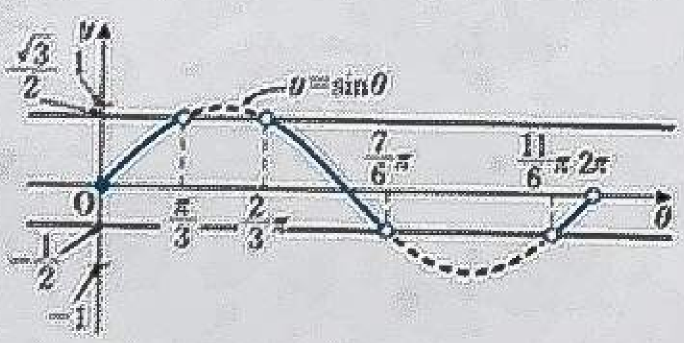
$\theta = \frac{\pi}{4},\ \frac{7}{4}\pi$

When $\cos\theta = \frac{\sqrt{3}}{2}$,

$\theta = \frac{\pi}{6},\ \frac{11}{6}\pi$

$\therefore\ \frac{\pi}{6} < \theta < \frac{\pi}{4},\ \frac{7}{4}\pi < \theta < \frac{11}{6}\pi$

4) $0 < 2\sin\theta + 1 < 1 + \sqrt{3}$



Sol] Rearranging,

$-\frac{1}{2} < \sin\theta < \frac{\sqrt{3}}{2}$

When $\sin\theta = -\frac{1}{2}$,

$\theta = \frac{7}{6}\pi,\ \frac{11}{6}\pi$

When $\sin\theta = \frac{\sqrt{3}}{2}$,

$\theta = \frac{\pi}{3},\ \frac{2}{3}\pi$

$\therefore\ 0 \leqq \theta < \frac{\pi}{3},\ \frac{2}{3}\pi < \theta < \frac{7}{6}\pi,\ \frac{11}{6}\pi < \theta < 2\pi$

Trigonometric Inequalities

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	1	2~

Given $0 \leqslant \theta < 2\pi$, solve the following inequalities.

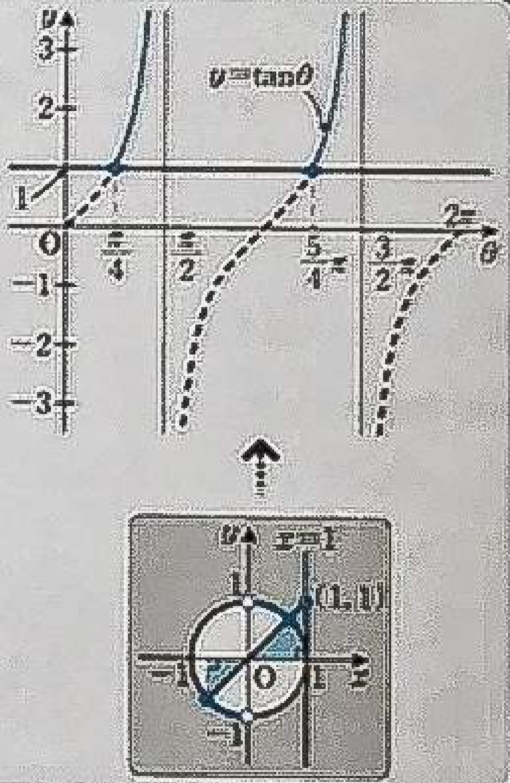
Ex. $\tan\theta \geqslant 1$

[Sol] When $\tan\theta = 1$,

$$\theta = \frac{\pi}{4},\ \frac{5}{4}\pi$$

Therefore,

$$\frac{\pi}{4} \leqslant \theta < \frac{\pi}{2},\ \frac{5}{4}\pi \leqslant \theta < \frac{3}{2}\pi$$



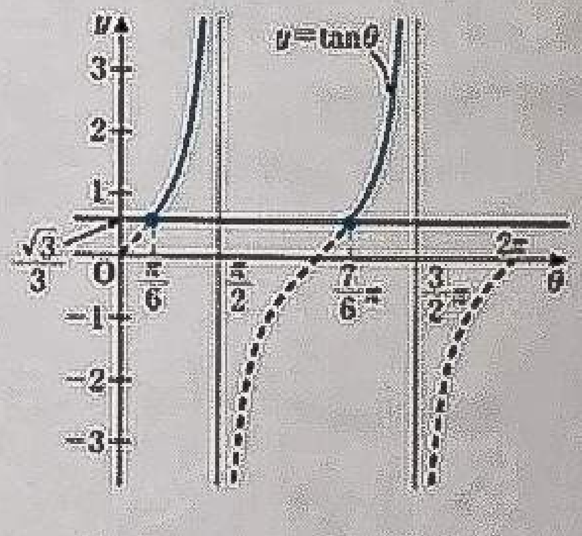
(1) $\tan\theta \geqslant \frac{\sqrt{3}}{3}$

[Sol] When $\tan\theta = \frac{\sqrt{3}}{3}$,

$$\theta = \frac{\pi}{6},\ \frac{7}{6}\pi$$

Therefore,

$$\frac{\pi}{6} \leqslant \theta < \frac{\pi}{2},\ \frac{7}{6}\pi \leqslant \theta < \frac{3}{2}\pi$$



M144b

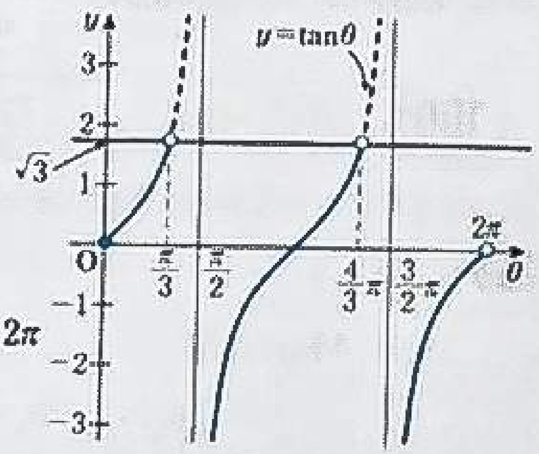
(2) $\tan\theta < \sqrt{3}$

[Sol] When $\tan\theta = \sqrt{3}$,

$$\theta = \frac{\pi}{3},\ \frac{4}{3}\pi$$

Therefore,

$$0 \leqq \theta < \frac{\pi}{3},\ \frac{\pi}{2} < \theta < \frac{4}{3}\pi,\ \frac{3}{2}\pi < \theta < 2\pi$$



(3) $-\sqrt{3} < \tan\theta < 1$

[Sol] When $\tan\theta = -\sqrt{3}$,

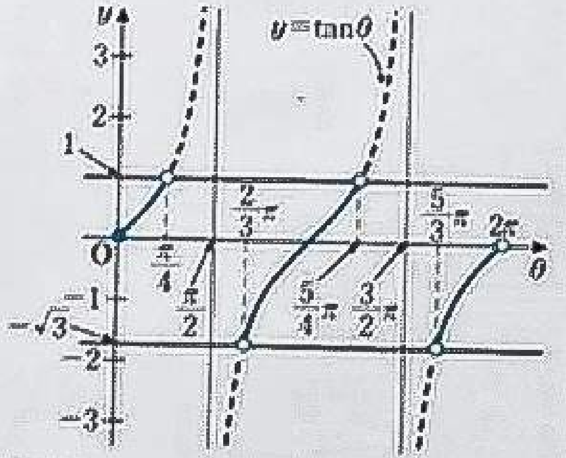
$$\theta = \frac{2}{3}\pi,\ \frac{5}{3}\pi$$

When $\tan\theta = 1$,

$$\theta = \frac{\pi}{4},\ \frac{5}{4}\pi$$

Therefore,

$$0 \leqq \theta < \frac{\pi}{4},\ \frac{2}{3}\pi < \theta < \frac{5}{4}\pi,\ \frac{5}{3}\pi < \theta < 2\pi$$



(4) $-2 < \sqrt{3}\tan\theta + 1 < 2$

[Sol] Rearranging,

$$-\sqrt{3} < \tan\theta < \frac{\sqrt{3}}{3}$$

When $\tan\theta = -\sqrt{3}$,

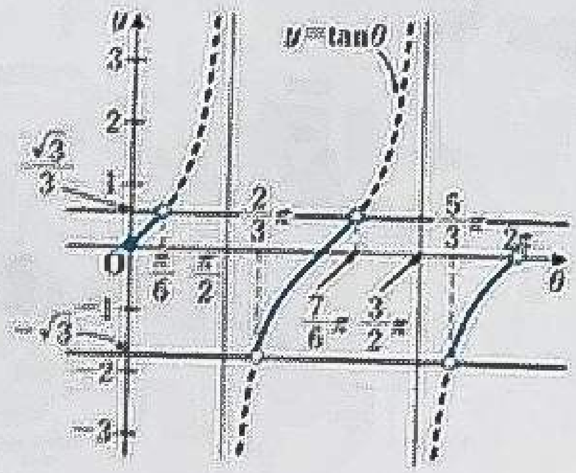
$$\theta = \frac{2}{3}\pi,\ \frac{5}{3}\pi$$

When $\tan\theta = \frac{\sqrt{3}}{3}$,

$$\theta = \frac{\pi}{6},\ \frac{7}{6}\pi$$

Therefore,

$$0 \leqq \theta < \frac{\pi}{6},\ \frac{2}{3}\pi < \theta < \frac{7}{6}\pi,\ \frac{5}{3}\pi < \theta < 2\pi$$



KUMON

Trigonometric Inequalities

Name

Date / /

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Time : to :

Given $0 \leqslant \theta < 2\pi$, solve the following inequalities.

Ex.

$\sin\left(\theta - \frac{\pi}{3}\right) > \frac{1}{2}$

[Sol] Since $0 \leqslant \theta < 2\pi$,

$$-\frac{\pi}{3} \leqslant \theta - \frac{\pi}{3} < \frac{5}{3}\pi$$

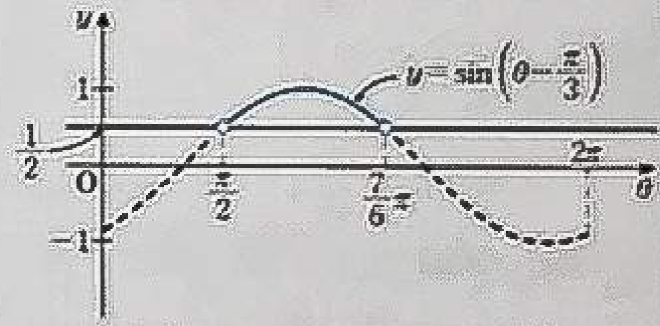
When $\sin\left(\theta - \frac{\pi}{3}\right) = \frac{1}{2}$,

$$\theta - \frac{\pi}{3} = \frac{\pi}{6},\ \frac{5}{6}\pi$$

$$\therefore\ \theta = \frac{\pi}{2},\ \frac{7}{6}\pi$$

Therefore,

$$\frac{\pi}{2} < \theta < \frac{7}{6}\pi$$



The graph of $y=\sin\left(\theta-\frac{\pi}{3}\right)$ is a translation of the graph of $y=\sin\theta$, $\frac{\pi}{3}$ units along the θ-axis.

(1) $\sin\left(\theta - \frac{\pi}{4}\right) > -\frac{1}{2}$

[Sol] Since $0 \leqslant \theta < 2\pi$,

$$-\frac{\pi}{4} \leqslant \theta - \frac{\pi}{4} < \frac{7}{4}\pi$$

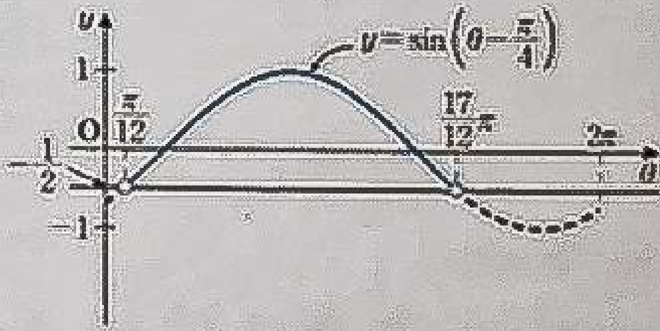
When $\sin\left(\theta - \frac{\pi}{4}\right) = -\frac{1}{2}$,

$$\theta - \frac{\pi}{4} = -\frac{\pi}{6},\ \frac{7}{6}\pi$$

$$\therefore\ \theta = \frac{\pi}{12},\ \frac{17}{12}\pi$$

Therefore,

$$\frac{\pi}{12} < \theta < \frac{17}{12}\pi$$

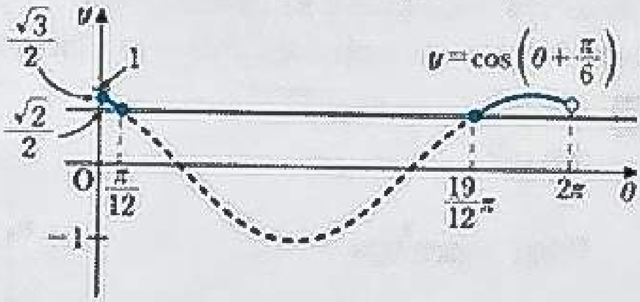


M145b

(2) $\cos\left(\theta+\frac{\pi}{6}\right) \geqq \frac{\sqrt{2}}{2}$

[Sol] Since $0 \leqq \theta < 2\pi$,

$$\frac{\pi}{6} \leqq \theta+\frac{\pi}{6} < \frac{13}{6}\pi$$



When $\cos\left(\theta+\frac{\pi}{6}\right)=\frac{\sqrt{2}}{2}$,

$$\theta+\frac{\pi}{6}=\frac{\pi}{4},\ \frac{7}{4}\pi$$

$$\therefore\ \theta=\frac{\pi}{12},\ \frac{19}{12}\pi$$

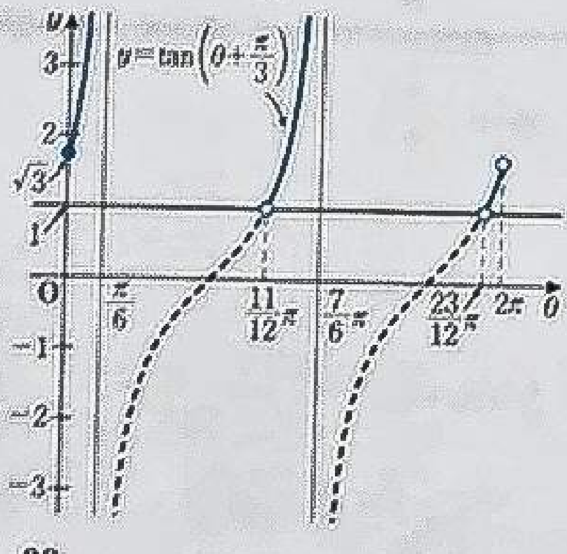
Therefore,

$$0 \leqq \theta \leqq \frac{\pi}{12},\ \frac{19}{12}\pi \leqq \theta < 2\pi$$

(3) $\tan\left(\theta+\frac{\pi}{3}\right) > 1$

[Sol] Since $0 \leqq \theta < 2\pi$,

$$\frac{\pi}{3} \leqq \theta+\frac{\pi}{3} < \frac{7}{3}\pi$$



When $\tan\left(\theta+\frac{\pi}{3}\right)=1$,

$$\theta+\frac{\pi}{3}=\frac{5}{4}\pi,\ \frac{9}{4}\pi$$

$$\therefore\ \theta=\frac{11}{12}\pi,\ \frac{23}{12}\pi$$

Therefore,

$$0 \leqq \theta < \frac{\pi}{6},\ \frac{11}{12}\pi < \theta < \frac{7}{6}\pi,\ \frac{23}{12}\pi < \theta < 2\pi$$

Trigonometric Inequalities

Name

Date / /

Time : to :

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Given $0 \leqslant \theta < 2\pi$, solve the following inequalities.

Ex. $2\cos^2\theta + 5\sin\theta - 4 > 0$

[Sol] $2(1-\sin^2\theta) + 5\sin\theta - 4 > 0$ ← $\cos^2\theta = 1 - \sin^2\theta$

$2\sin^2\theta - 5\sin\theta + 2 < 0$

$(2\sin\theta - 1)(\sin\theta - 2) < 0$

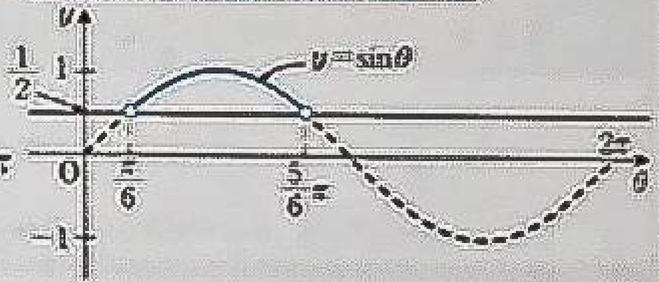
Since $0 \leqslant \theta < 2\pi$, $-1 \leqslant \sin\theta \leqslant 1$

Since $-1 \leqslant \sin\theta \leqslant 1$, always $\sin\theta - 2 < 0$.

$\therefore 2\sin\theta - 1 > 0$ ← When $AB < 0$ and $B < 0$, $A > 0$

$\therefore \sin\theta > \frac{1}{2}$

When $\sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}, \frac{5}{6}\pi$



Therefore,

$\frac{\pi}{6} < \theta < \frac{5}{6}\pi$

(1) $2\sin^2\theta < 3\cos\theta$

[Sol] $2(1-\cos^2\theta) < 3\cos\theta$

$2\cos^2\theta + 3\cos\theta - 2 > 0$

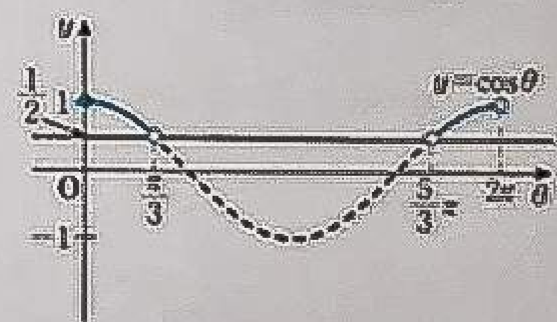
$(2\cos\theta - 1)(\cos\theta + 2) > 0$

Since $-1 \leqslant \cos\theta \leqslant 1$, always $\cos\theta + 2 > 0$.

$\therefore 2\cos\theta - 1 > 0$

$\therefore \cos\theta > \frac{1}{2}$

When $\cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}, \frac{5}{3}\pi$



Therefore,

$0 \leqslant \theta < \frac{\pi}{3}, \frac{5}{3}\pi < \theta < 2\pi$

M146b

(2) $2\cos^2\theta + \sin\theta - 1 > 0$

[Sol] $2(1 - \sin^2\theta) + \sin\theta - 1 > 0$

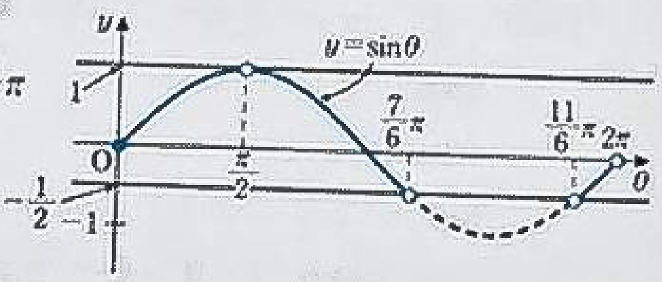
$2\sin^2\theta - \sin\theta - 1 < 0$

$(2\sin\theta + 1)(\sin\theta - 1) < 0$

$\therefore\ -\frac{1}{2} < \sin\theta < 1$

When $\sin\theta = -\frac{1}{2}$, $\theta = \frac{7}{6}\pi$, $\frac{11}{6}\pi$

When $\sin\theta = 1$, $\theta = \frac{\pi}{2}$



Therefore,

$0 \leqq \theta < \frac{\pi}{2}$, $\frac{\pi}{2} < \theta < \frac{7}{6}\pi$, $\frac{11}{6}\pi < \theta < 2\pi$

(3) $\sqrt{3}\tan^2\theta + 3\tan\theta > 0$

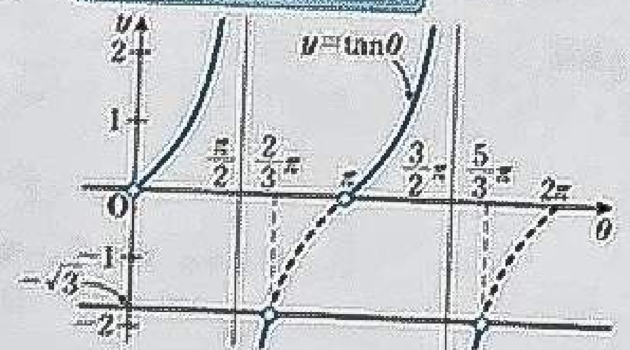
[Sol] $\tan\theta(\sqrt{3}\tan\theta + 3) > 0$

$\therefore\ \tan\theta < -\sqrt{3}$, $0 < \tan\theta$

For $0 \leqq \theta < 2\pi$,
$\tan\theta$ takes all real values.

When $\tan\theta = -\sqrt{3}$, $\theta = \frac{2}{3}\pi$, $\frac{5}{3}\pi$

When $\tan\theta = 0$, $\theta = 0, \pi$



Therefore,

$0 < \theta < \frac{\pi}{2}$, $\frac{\pi}{2} < \theta < \frac{2}{3}\pi$, $\pi < \theta < \frac{3}{2}\pi$, $\frac{3}{2}\pi < \theta < \frac{5}{3}\pi$

Trigonometric inequalities

Name

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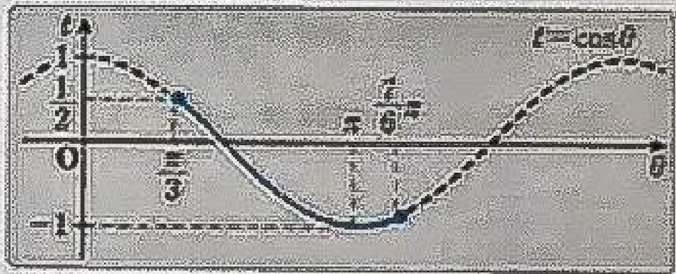
Find the maximum and minimum values of each given function and state the corresponding values of θ.

Ex

$y=2\cos\theta-3 \quad \left(\frac{\pi}{3} \leqslant \theta \leqslant \frac{7}{6}\pi\right)$

[Sol] Let $\cos\theta=t$.

$-1 \leqslant t \leqslant \frac{1}{2}$ ←



$y=2t-3$

Therefore,

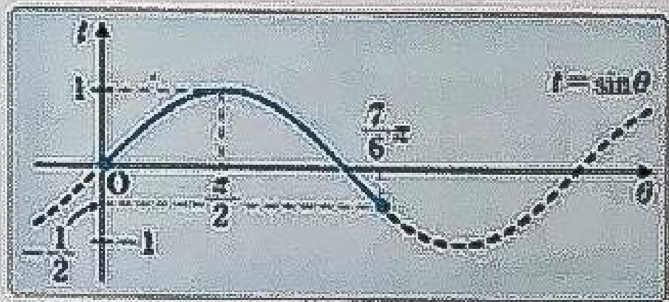
the maximum value is -2, at $t=\cos\theta=\frac{1}{2}$, i.e. $\theta=\frac{\pi}{3}$ and

the minimum value is -5, at $t=\cos\theta=-1$, i.e. $\theta=\pi$.

(1) $y=-2\sin\theta+3 \quad \left(0 \leqslant \theta \leqslant \frac{7}{6}\pi\right)$

[Sol] Let $\sin\theta=t$.

$-\frac{1}{2} \leqslant t \leqslant 1$ ←



$y=-2t+3$

Therefore,

the maximum value is 4, at $t=\sin\theta=-\frac{1}{2}$, i.e. $\theta=\frac{7}{6}\pi$ and

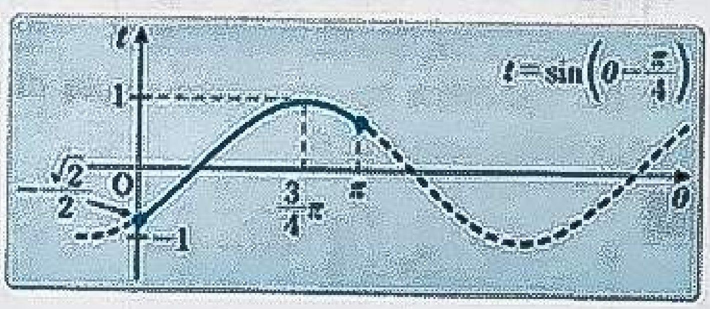
the minimum value is 1, at $t=\sin\theta=1$, i.e. $\theta=\frac{\pi}{2}$.

M147b

(2) $y=2\sin\left(\theta-\frac{\pi}{4}\right)-1 \quad (0\leqq\theta\leqq\pi)$

[Sol] Let $\sin\left(\theta-\frac{\pi}{4}\right)=t$.

$-\frac{\sqrt{2}}{2}\leqq t\leqq 1$ ⇐



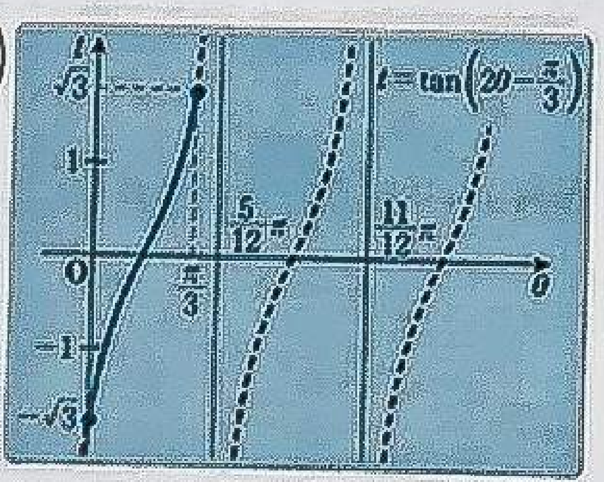
$y=2t-1$

Therefore,

the maximum value is 1, at $t=\sin\left(\theta-\frac{\pi}{4}\right)=1$, i.e. $\theta=\frac{3}{4}\pi$ and

the minimum value is $-\sqrt{2}-1$, at $t=\sin\left(\theta-\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$, i.e. $\theta=0$.

(3) $y=\tan\left(2\theta-\frac{\pi}{3}\right)+1 \quad \left(0\leqq\theta\leqq\frac{\pi}{3}\right)$



[Sol] Let $\tan\left(2\theta-\frac{\pi}{3}\right)=t$.

$-\sqrt{3}\leqq t\leqq\sqrt{3}$ ⇐

$y=t+1$

Therefore,

the maximum value is $\sqrt{3}+1$, at $t=\tan\left(2\theta-\frac{\pi}{3}\right)=\sqrt{3}$, i.e. $\theta=\frac{\pi}{3}$ and

the minimum value is $-\sqrt{3}+1$, at $t=\tan\left(2\theta-\frac{\pi}{3}\right)=-\sqrt{3}$, i.e. $\theta=0$.

Trigonometric Inequalities

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Given $0 \leqslant \theta < 2\pi$, find the maximum and minimum values of each given function and state the corresponding values of θ.

Ex. $y=\sin^2\theta+\sin\theta+3$

[Sol] Let $\sin\theta=t$.

$-1 \leqslant t \leqslant 1$ ← Since $0 \leqslant \theta < 2\pi$, $-1 \leqslant \sin\theta \leqslant 1$

$y=t^2+t+3$

$=\left(t+\frac{1}{2}\right)^2+\frac{11}{4}$

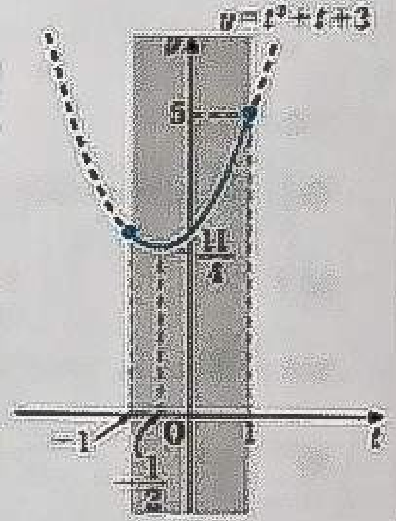
Therefore,

the maximum value is 5,

at $t=\sin\theta=1$, i.e. $\theta=\frac{\pi}{2}$ and

the minimum value is $\frac{11}{4}$,

at $t=\sin\theta=-\frac{1}{2}$, i.e. $\theta=\frac{7}{6}\pi$, $\frac{11}{6}\pi$.



(1) $y=\cos^2\theta+\cos\theta-1$

[Sol] Let $\cos\theta=t$.

$-1 \leqslant t \leqslant 1$

$y=t^2+t-1$

$=\left(t+\frac{1}{2}\right)^2-\frac{5}{4}$

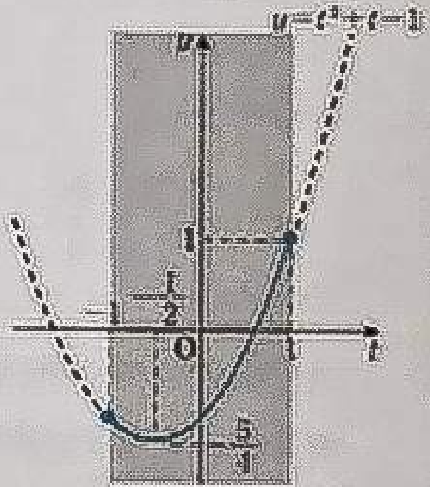
Therefore,

the maximum value is 1,

at $t=\cos\theta=1$, i.e. $\theta=0$ and

the minimum value is $-\frac{5}{4}$,

at $t=\cos\theta=-\frac{1}{2}$, i.e. $\theta=\frac{2}{3}\pi$, $\frac{4}{3}\pi$.



M I 48b

(2) $y=\sin\theta+\cos^2\theta$

$\boxed{\cos^2\theta=1-\sin^2\theta}$

[Sol] $y=\sin\theta+(1-\sin^2\theta)$

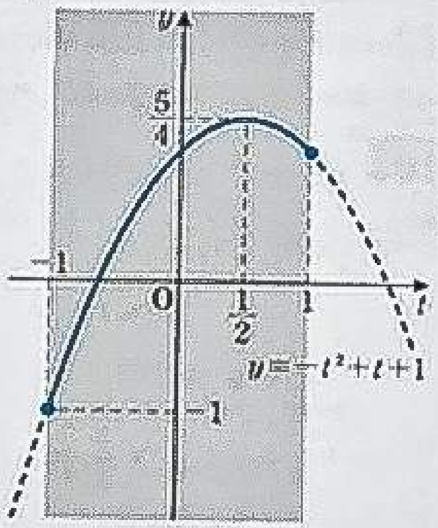
$=-\sin^2\theta+\sin\theta+1$

Let $\sin\theta=t$.

$-1\leqq t\leqq 1$

$y=-t^2+t+1$

$=-\left(t-\frac{1}{2}\right)^2+\frac{5}{4}$



Therefore,

the maximum value is $\frac{5}{4}$,

at $t=\sin\theta=\frac{1}{2}$, i.e. $\theta=\frac{\pi}{6}$, $\frac{5}{6}\pi$ and

the minimum value is -1,

at $t=\sin\theta=-1$, i.e. $\theta=\frac{3}{2}\pi$.

(3) $y=3\cos\theta-\sin^2\theta$

[Sol] $y=3\cos\theta-(1-\cos^2\theta)$

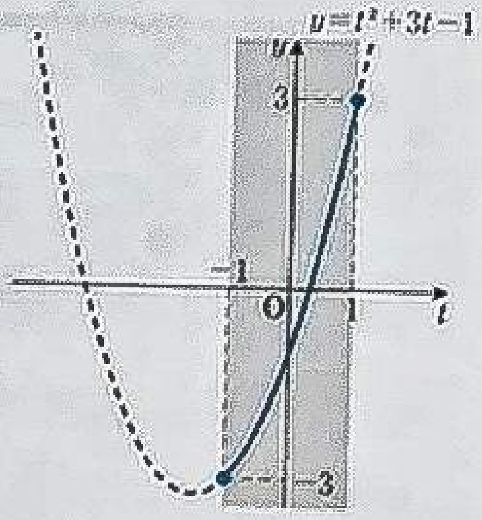
$=\cos^2\theta+3\cos\theta-1$

Let $\cos\theta=t$.

$-1\leqq t\leqq 1$

$y=t^2+3t-1$

$=\left(t+\frac{3}{2}\right)^2-\frac{13}{4}$



Therefore,

the maximum value is 3,

at $t=\cos\theta=1$, i.e. $\theta=0$ and

the minimum value is -3,

at $t=\cos\theta=-1$, i.e. $\theta=\pi$.

Trigonometric Inequalities

Name

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1. Let α be the minimum value of θ satisfying $5\cos^2\theta+13\sin\theta-11\geqq 0$ for $0\leqq\theta\leqq\frac{\pi}{2}$. Find the values of $\sin\alpha$ and $\tan\alpha$.

[Sol] Since $5(1-\sin^2\theta)+13\sin\theta-11\geqq 0$,

$$5\sin^2\theta-13\sin\theta+6\leqq 0$$

Therefore,

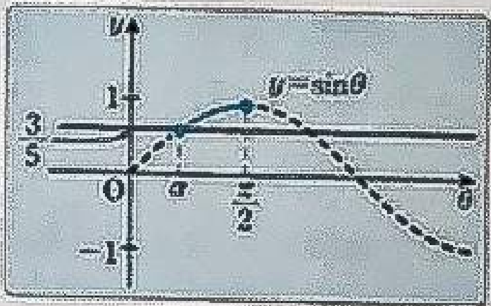
$$(5\sin\theta-3)(\sin\theta-2)\leqq 0$$

Since $0\leqq\sin\theta\leqq 1$, always $\sin\theta-2<0$. ← Since $0\leqq\theta\leqq\frac{\pi}{2}$, $0\leqq\sin\theta\leqq 1$

$$\therefore\ 5\sin\theta-3\geqq 0$$

$$\therefore\ \sin\theta\geqq\frac{3}{5}$$

$$\therefore\ \sin\alpha=\frac{3}{5}$$



Since $0\leqq\alpha\leqq\frac{\pi}{2}$, $\cos\alpha\geqq 0$

$$\cos\alpha=\sqrt{1-\left(\frac{3}{5}\right)^2}$$

$$=\frac{4}{5}$$

$$\therefore\ \tan\alpha=\frac{3}{5}\div\frac{4}{5}$$

$$=\frac{3}{4}$$

M149b

2. Given $0 \leqslant \theta < 2\pi$, find the range of values of θ for which two parabolas $y=2\sqrt{3}(x-\cos\theta)^2+\sin\theta$ and $y=-2\sqrt{3}(x+\cos\theta)^2-\sin\theta$ intersect at two different points.

[Sol] $y=2\sqrt{3}(x-\cos\theta)^2+\sin\theta \quad \cdots ①$

$y=-2\sqrt{3}(x+\cos\theta)^2-\sin\theta \quad \cdots ②$

From ① and ②,

$$2\sqrt{3}(x-\cos\theta)^2+\sin\theta=-2\sqrt{3}(x+\cos\theta)^2-\sin\theta$$

$$2\sqrt{3}x^2+2\sqrt{3}\cos^2\theta+\sin\theta=0$$

$$\frac{D}{4}=0^2-2\sqrt{3}(2\sqrt{3}\cos^2\theta+\sin\theta)>0$$

When $\frac{D}{4}>0$, $2\sqrt{3}x^2+2\sqrt{3}\cos^2\theta+\sin\theta=0$ has two different real solutions. (K76)

$$2\sqrt{3}\cos^2\theta+\sin\theta<0$$

$$2\sqrt{3}(1-\sin^2\theta)+\sin\theta<0$$

$$2\sqrt{3}\sin^2\theta-\sin\theta-2\sqrt{3}>0$$

$$(2\sin\theta+\sqrt{3})(\sqrt{3}\sin\theta-2)>0$$

Since $-1 \leqslant \sin\theta \leqslant 1$, always $\sqrt{3}\sin\theta-2<0$.

Since $0 \leqslant \theta < 2\pi$, $-1 \leqslant \sin\theta \leqslant 1$

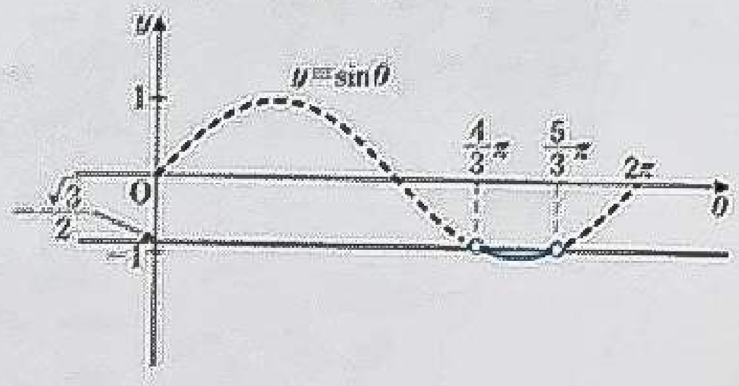
$$\therefore\ 2\sin\theta+\sqrt{3}<0$$

$$\sin\theta<-\frac{\sqrt{3}}{2}$$

When $\sin\theta=-\frac{\sqrt{3}}{2}$, $\theta=\frac{4}{3}\pi, \frac{5}{3}\pi$

Therefore,

$$\frac{4}{3}\pi<\theta<\frac{5}{3}\pi$$



Trigonometric Inequalities

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1. Given $0 \leqq \theta < 2\pi$, solve the following inequalities.

(1) $\sin\theta > \dfrac{\sqrt{3}}{2}$

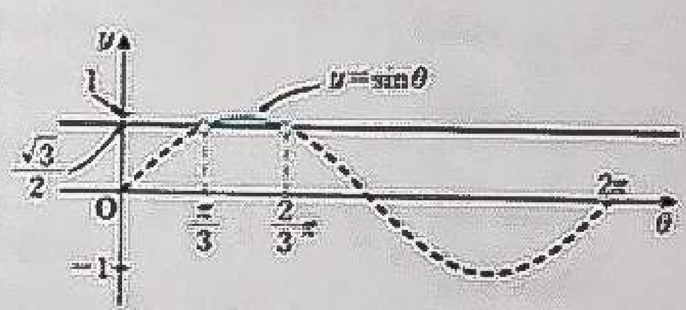
➡ M141

[Sol] When $\sin\theta = \dfrac{\sqrt{3}}{2}$,

$\theta = \dfrac{\pi}{3}, \dfrac{2}{3}\pi$

Therefore,

$\dfrac{\pi}{3} < \theta < \dfrac{2}{3}\pi$



(2) $\cos\theta > \dfrac{1}{2}$

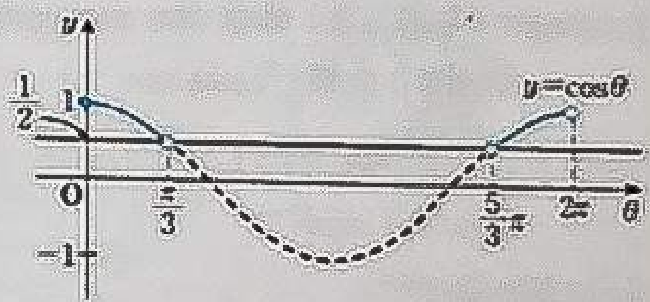
➡ M142

[Sol] When $\cos\theta = \dfrac{1}{2}$,

$\theta = \dfrac{\pi}{3}, \dfrac{5}{3}\pi$

Therefore,

$0 \leqq \theta < \dfrac{\pi}{3}, \dfrac{5}{3}\pi < \theta < 2\pi$



(3) $-\dfrac{\sqrt{3}}{3} < \tan\theta < 1$

➡ M144

[Sol] When $\tan\theta = -\dfrac{\sqrt{3}}{3}$,

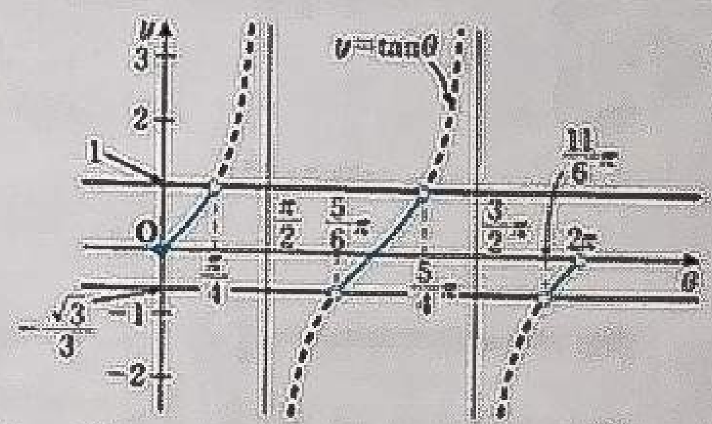
$\theta = \dfrac{5}{6}\pi, \dfrac{11}{6}\pi$

When $\tan\theta = 1$,

$\theta = \dfrac{\pi}{4}, \dfrac{5}{4}\pi$

Therefore,

$0 \leqq \theta < \dfrac{\pi}{4}, \dfrac{5}{6}\pi < \theta < \dfrac{5}{4}\pi, \dfrac{11}{6}\pi < \theta < 2\pi$



M150b

(4) $3\sin\theta - 2\cos^2\theta > 0$ ➡ M146

[Sol] $3\sin\theta - 2(1-\sin^2\theta) > 0$

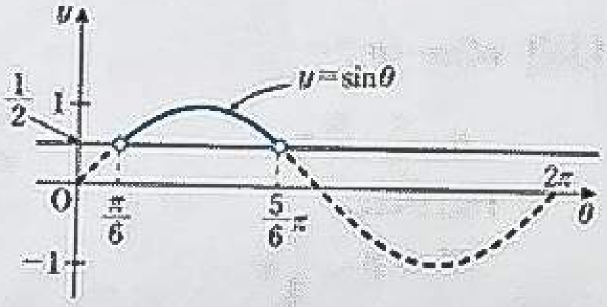
$2\sin^2\theta + 3\sin\theta - 2 > 0$

$(2\sin\theta - 1)(\sin\theta + 2) > 0$

Since $-1 \leqslant \sin\theta \leqslant 1$, always $\sin\theta + 2 > 0$.

$\therefore\ 2\sin\theta - 1 > 0$

$\therefore\ \sin\theta > \frac{1}{2}$



When $\sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}, \frac{5}{6}\pi$

Therefore,

$\frac{\pi}{6} < \theta < \frac{5}{6}\pi$

2. Given $0 \leqslant \theta < 2\pi$, find the maximum and minimum values of function $y = -\cos^2\theta - \sin\theta + 3$ and state the corresponding values of θ. ➡ M148

[Sol] $y = -(1-\sin^2\theta) - \sin\theta + 3$

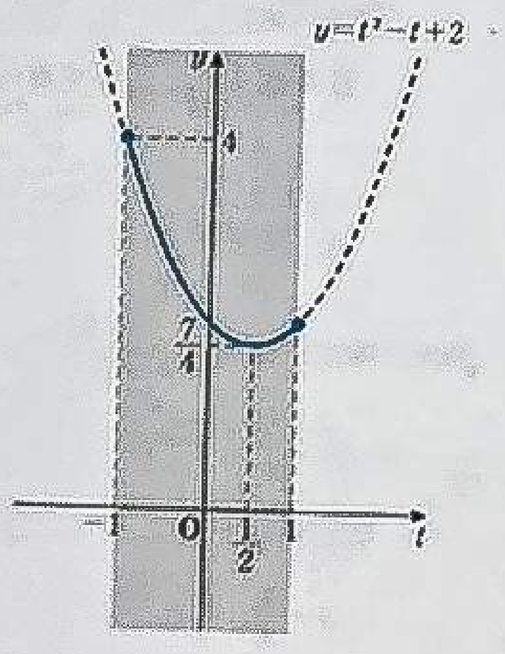
$= \sin^2\theta - \sin\theta + 2$

Let $\sin\theta = t$.

$-1 \leqslant t \leqslant 1$

$y = t^2 - t + 2$

$= \left(t - \frac{1}{2}\right)^2 + \frac{7}{4}$



Therefore,

the maximum value is 4,

at $t = \sin\theta = -1$, i.e. $\theta = \frac{3}{2}\pi$ and

the minimum value is $\frac{7}{4}$,

at $t = \sin\theta = \frac{1}{2}$, i.e. $\theta = \frac{\pi}{6}, \frac{5}{6}\pi$.

Addition Formulas 1

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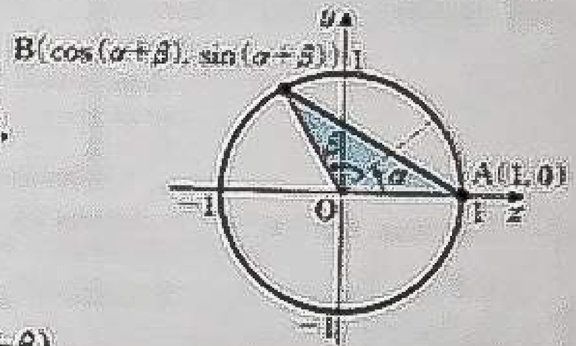
Prove the identity $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$.

[Sol] Given that point B is the point of intersection of the unit circle and the terminal side forming the angle $\alpha+\beta$, place point $A(1, 0)$ and point

$B(\cos(\alpha+\beta), \sin(\alpha+\beta))$

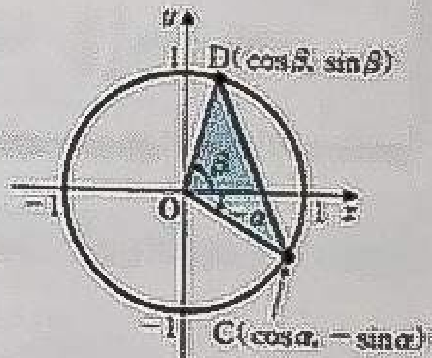
as shown in the diagram on the right.



$AB^2=[\cos(\alpha+\beta)-1]^2+\sin^2(\alpha+\beta)$

$=2-2\boxed{\cos(\alpha+\beta)}$ ← $\sin^2(\alpha+\beta)+\cos^2(\alpha+\beta)=1$

Next, let points C and D be the points obtained by rotating two points A and B by $-\alpha$ about the origin.



$C(\cos\alpha, -\sin\alpha)$ ← $\cos(-\theta)=\cos\theta$, $\sin(-\theta)=-\sin\theta$

$D(\cos\beta, \sin\beta)$

$CD^2=(\cos\beta-\cos\alpha)^2+(\sin\beta+\sin\alpha)^2$

$=2-2(\boxed{\cos\alpha\cos\beta-\sin\alpha\sin\beta})$

Since $AB^2=CD^2$, ← Since C and D are the points obtained by rotating two points A and B, AB=CD

$2-2\boxed{\cos(\alpha+\beta)}=2-2(\boxed{\cos\alpha\cos\beta-\sin\alpha\sin\beta})$

Therefore,

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ is true.

Answers: $\cos(\alpha+\beta)$, $\cos\alpha\cos\beta-\sin\alpha\sin\beta$, $\cos(\alpha+\beta)$, $\cos\alpha\cos\beta-\sin\alpha\sin\beta$

From the above, the following identity is true for any α and β.

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

M151b

Using the identity $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ …①, solve the following questions.

(1) Express $\cos(\alpha-\beta)$ in terms of $\cos\alpha$, $\cos\beta$, $\sin\alpha$ and $\sin\beta$ by substituting $(-\beta)$ for β on both sides of identity ①.

[Sol] $\cos[\alpha+(-\beta)]=\cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta)$

$\therefore\ \cos(\alpha-\beta)=\boxed{\cos\alpha\cos\beta+\sin\alpha\sin\beta}$

$\cos(-\theta)=\cos\theta$
$\sin(-\theta)=-\sin\theta$

(2) Express $\sin(\alpha-\beta)$ in terms of $\sin\alpha$, $\sin\beta$, $\cos\alpha$ and $\cos\beta$ by substituting $\left(\frac{\pi}{2}-\alpha\right)$ for α on both sides of identity ①.

[Sol] $\cos\left[\left(\frac{\pi}{2}-\alpha\right)+\beta\right]=\cos\left(\frac{\pi}{2}-\alpha\right)\cos\beta-\sin\left(\frac{\pi}{2}-\alpha\right)\sin\beta$

LHS$=\cos\left[\frac{\pi}{2}-(\alpha-\beta)\right]=\sin(\alpha-\beta)$

RHS$=\boxed{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$

$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$
$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$

$\therefore\ \sin(\alpha-\beta)=\boxed{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$

(3) Express $\sin(\alpha+\beta)$ in terms of $\sin\alpha$, $\sin\beta$, $\cos\alpha$ and $\cos\beta$ by substituting $\left(\frac{\pi}{2}-\alpha\right)$ for α and $(-\beta)$ for β on both sides of identity ①.

[Sol] $\cos\left[\left(\frac{\pi}{2}-\alpha\right)+(-\beta)\right]=\cos\left(\frac{\pi}{2}-\alpha\right)\cos(-\beta)-\sin\left(\frac{\pi}{2}-\alpha\right)\sin(-\beta)$

LHS$=\cos\left[\frac{\pi}{2}-(\alpha+\beta)\right]=\sin(\alpha+\beta)$

RHS$=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\therefore\ \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

The identity $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ and the identities obtained in (1)~(3) are called the ***Addition Formulas***.

M152a

KUMON

Addition Formulas 1

Name

Date / /

Time : to :

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0		1	2	3

Addition Formulas I

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Evaluate the following expressions using the formulas above.

Ex.

$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ+30^\circ)\\ &= \sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

(1)

$$\begin{aligned}\cos 75^\circ &= \cos(45^\circ+30^\circ)\\ &= \cos 45^\circ\cos 30^\circ-\sin 45^\circ\sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

(2)

$$\begin{aligned}\sin 105^\circ &= \sin(60^\circ+45^\circ)\\ &= \sin 60^\circ\cos 45^\circ+\cos 60^\circ\sin 45^\circ\\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Alternative Solution

$$\begin{aligned}\sin 105^\circ &= \sin(135^\circ-30^\circ)\\ &= \sin 135^\circ\cos 30^\circ-\cos 135^\circ\sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

(3)

$$\begin{aligned}\cos 105^\circ &= \cos(60^\circ+45^\circ)\\ &= \cos 60^\circ\cos 45^\circ-\sin 60^\circ\sin 45^\circ\\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

Alternative Solution

$$\begin{aligned}\cos 105^\circ &= \cos(135^\circ-30^\circ)\\ &= \cos 135^\circ\cos 30^\circ+\sin 135^\circ\sin 30^\circ\\ &= -\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= -\frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

M152b

(4) $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$ $\boxed{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$

$= \dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{2}}{2}\cdot\dfrac{1}{2}$

$= \dfrac{\sqrt{6}-\sqrt{2}}{4}$

(5) $\cos 15^\circ = \cos(45^\circ - 30^\circ)$

$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$ $\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}$

$= \dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{2}}{2}\cdot\dfrac{1}{2}$

$= \dfrac{\sqrt{6}+\sqrt{2}}{4}$

[Alternative Solution

$\cos 15^\circ = \cos(60^\circ - 45^\circ)$

$= \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ$

$= \dfrac{1}{2}\cdot\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{2}}{2}$

$= \dfrac{\sqrt{2}+\sqrt{6}}{4}$]

(6) $\sin 165^\circ = \sin(120^\circ + 45^\circ)$

$= \sin 120^\circ \cos 45^\circ + \cos 120^\circ \sin 45^\circ$

$= \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{2}}{2} + \left(-\dfrac{1}{2}\right)\cdot\dfrac{\sqrt{2}}{2}$

$= \dfrac{\sqrt{6}-\sqrt{2}}{4}$

[Alternative Solution

$\sin 165^\circ = \sin(135^\circ + 30^\circ)$

$= \sin 135^\circ \cos 30^\circ + \cos 135^\circ \sin 30^\circ$

$= \dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{3}}{2} + \left(-\dfrac{\sqrt{2}}{2}\right)\cdot\dfrac{1}{2}$

$= \dfrac{\sqrt{6}-\sqrt{2}}{4}$]

(7) $\cos 165^\circ = \cos(120^\circ + 45^\circ)$

$= \cos 120^\circ \cos 45^\circ - \sin 120^\circ \sin 45^\circ$

$= -\dfrac{1}{2}\cdot\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{2}}{2}$

$= -\dfrac{\sqrt{2}+\sqrt{6}}{4}$

[Alternative Solution

$\cos 165^\circ = \cos(135^\circ + 30^\circ)$

$= \cos 135^\circ \cos 30^\circ - \sin 135^\circ \sin 30^\circ$

$= -\dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{2}}{2}\cdot\dfrac{1}{2}$

$= -\dfrac{\sqrt{6}+\sqrt{2}}{4}$]

Addition Formulas 1

100%	~90%	~80%	~70%	69%~

Name

Date / /

Time : to :

Ex.

Let $\frac{\pi}{2}<\alpha<\pi$ and $0<\beta<\frac{\pi}{2}$. Find the value of $\sin(\alpha+\beta)$ when $\sin\alpha=\frac{4}{5}$ and $\cos\beta=\frac{5}{13}$.

[Sol] Since $\frac{\pi}{2}<\alpha<\pi$, $\cos\alpha<0$

$$\cos\alpha=-\sqrt{1-\left(\frac{4}{5}\right)^2}=-\frac{3}{5}$$

← Since $\cos\alpha<0$, $\cos\alpha=-\sqrt{1-\sin^2\alpha}$

Also, since $0<\beta<\frac{\pi}{2}$, $\sin\beta>0$

$$\sin\beta=\sqrt{1-\left(\frac{5}{13}\right)^2}=\frac{12}{13}$$

← Since $\sin\beta>0$, $\sin\beta=\sqrt{1-\cos^2\beta}$

Therefore,

$$\sin(\alpha+\beta)=\frac{4}{5}\cdot\frac{5}{13}+\left(-\frac{3}{5}\right)\cdot\frac{12}{13}$$
$$=-\frac{16}{65}$$

← $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

1. Let $0<\alpha<\frac{\pi}{2}$ and $\frac{\pi}{2}<\beta<\pi$. Find the value of $\cos(\alpha+\beta)$ when $\sin\alpha=\frac{3}{5}$ and $\sin\beta=\frac{5}{13}$.

[Sol] Since $0<\alpha<\frac{\pi}{2}$, $\cos\alpha>0$

$$\cos\alpha=\sqrt{1-\left(\frac{3}{5}\right)^2}=\frac{4}{5}$$

Also, since $\frac{\pi}{2}<\beta<\pi$, $\cos\beta<0$

$$\cos\beta=-\sqrt{1-\left(\frac{5}{13}\right)^2}=-\frac{12}{13}$$

Therefore,

$$\cos(\alpha+\beta)=\frac{4}{5}\cdot\left(-\frac{12}{13}\right)-\frac{3}{5}\cdot\frac{5}{13}$$
$$=-\frac{63}{65}$$

M153b

2. Let $\frac{\pi}{2} < \alpha < \pi$ and $\pi < \beta < \frac{3}{2}\pi$. Find the value of $\sin(\alpha - \beta)$ when $\sin\alpha = \frac{3}{5}$ and $\cos\beta = -\frac{12}{13}$.

[Sol] Since $\frac{\pi}{2} < \alpha < \pi$, $\cos\alpha < 0$

$$\cos\alpha = -\sqrt{1-\left(\frac{3}{5}\right)^2} = -\frac{4}{5}$$

Also, since $\pi < \beta < \frac{3}{2}\pi$, $\sin\beta < 0$

$$\sin\beta = -\sqrt{1-\left(-\frac{12}{13}\right)^2} = -\frac{5}{13}$$

Therefore,

$$\sin(\alpha-\beta) = \frac{3}{5}\cdot\left(-\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\cdot\left(-\frac{5}{13}\right)$$
$$= -\frac{56}{65}$$

3. Let $\pi < \alpha < \frac{3}{2}\pi$ and $\frac{3}{2}\pi < \beta < 2\pi$. Find the value of $\cos(\alpha - \beta)$ when $\cos\alpha = -\frac{4}{5}$ and $\cos\beta = \frac{8}{17}$.

[Sol] Since $\pi < \alpha < \frac{3}{2}\pi$, $\sin\alpha < 0$

$$\sin\alpha = -\sqrt{1-\left(-\frac{4}{5}\right)^2} = -\frac{3}{5}$$

Also, since $\frac{3}{2}\pi < \beta < 2\pi$, $\sin\beta < 0$

$$\sin\beta = -\sqrt{1-\left(\frac{8}{17}\right)^2} = -\frac{15}{17}$$

Therefore,

$$\cos(\alpha-\beta) = -\frac{4}{5}\cdot\frac{8}{17} + \left(-\frac{3}{5}\right)\cdot\left(-\frac{15}{17}\right)$$
$$= \frac{13}{85}$$

Addition Formulas 1

Name

Date / /

Time : to :

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1. Let $\frac{\pi}{2}<\alpha<\pi$ and $\frac{\pi}{2}<\beta<\pi$. Find the value of $\sin(\alpha+\beta)$ when $\tan\alpha=-\frac{3}{4}$ and $\cos\beta=-\frac{2\sqrt{5}}{5}$.

[Sol] $\frac{1}{\cos^2\alpha}=1+\left(-\frac{3}{4}\right)^2=\frac{25}{16}$ $\therefore\ \cos^2\alpha=\frac{16}{25}$ ← $\boxed{\frac{1}{\cos^2\alpha}=1+\tan^2\alpha}$

Since $\frac{\pi}{2}<\alpha<\pi$, $\cos\alpha<0$

$\cos\alpha=-\frac{4}{5}$, $\sin\alpha=-\frac{3}{4}\cdot\left(-\frac{4}{5}\right)=\frac{3}{5}$ ← $\boxed{\sin\alpha=\tan\alpha\cos\alpha}$

Also, since $\frac{\pi}{2}<\beta<\pi$, $\sin\beta>0$

$\sin\beta=\sqrt{1-\left(-\frac{2\sqrt{5}}{5}\right)^2}=\frac{\sqrt{5}}{5}$

Therefore,

$\sin(\alpha+\beta)=\frac{3}{5}\cdot\left(-\frac{2\sqrt{5}}{5}\right)+\left(-\frac{4}{5}\right)\cdot\frac{\sqrt{5}}{5}$ ← $\boxed{\begin{array}{l}\sin(\alpha+\beta)\\=\sin\alpha\cos\beta+\cos\alpha\sin\beta\end{array}}$

$=-\frac{2\sqrt{5}}{5}$

2. Let $0<\alpha<\frac{\pi}{2}$ and $\frac{\pi}{2}<\beta<\pi$. Find the value of $\cos(\alpha-\beta)$ when $\tan\alpha=1$ and $\tan\beta=-2$.

[Sol] $\frac{1}{\cos^2\alpha}=1+1^2=2$ $\therefore\ \cos^2\alpha=\frac{1}{2}$

Since $0<\alpha<\frac{\pi}{2}$, $\cos\alpha>0$

$\cos\alpha=\frac{\sqrt{2}}{2}$, $\sin\alpha=1\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}$

Also, $\frac{1}{\cos^2\beta}=1+(-2)^2=5$ $\therefore\ \cos^2\beta=\frac{1}{5}$

Since $\frac{\pi}{2}<\beta<\pi$, $\cos\beta<0$

$\cos\beta=-\frac{\sqrt{5}}{5}$, $\sin\beta=-2\cdot\left(-\frac{\sqrt{5}}{5}\right)=\frac{2\sqrt{5}}{5}$

Therefore,

$\cos(\alpha-\beta)=\frac{\sqrt{2}}{2}\cdot\left(-\frac{\sqrt{5}}{5}\right)+\frac{\sqrt{2}}{2}\cdot\frac{2\sqrt{5}}{5}$ ← $\boxed{\begin{array}{l}\cos(\alpha-\beta)\\=\cos\alpha\cos\beta+\sin\alpha\sin\beta\end{array}}$

$=\frac{\sqrt{10}}{10}$

M154b

EX

Find the value of $\sin(\alpha+\beta)$ when $\sin\alpha+\cos\beta=\frac{5}{4}$ …① and $\cos\alpha+\sin\beta=\frac{5}{4}$ …②.

[Sol] From ①, $\sin^2\alpha+2\sin\alpha\cos\beta+\cos^2\beta=\frac{25}{16}$ …③

From ②, $\cos^2\alpha+2\cos\alpha\sin\beta+\sin^2\beta=\frac{25}{16}$ …④ $\boxed{\sin^2\theta+\cos^2\theta=1}$

From ③+④, $2+2(\sin\alpha\cos\beta+\cos\alpha\sin\beta)=\frac{25}{8}$

$\therefore\ \sin\alpha\cos\beta+\cos\alpha\sin\beta=\frac{9}{16}$

$\therefore\ \sin(\alpha+\beta)=\frac{9}{16}$ $\boxed{\sin\alpha\cos\beta+\cos\alpha\sin\beta=\sin(\alpha+\beta)}$

3. Find the value of $\cos(\alpha+\beta)$ when $\sin\alpha-\sin\beta=\frac{1}{2}$ …① and $\cos\alpha+\cos\beta=\frac{1}{2}$ …②.

[Sol] From ①, $\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta=\frac{1}{4}$ …③

From ②, $\cos^2\alpha+2\cos\alpha\cos\beta+\cos^2\beta=\frac{1}{4}$ …④

From ③+④, $2+2(\cos\alpha\cos\beta-\sin\alpha\sin\beta)=\frac{1}{2}$

$\therefore\ \cos\alpha\cos\beta-\sin\alpha\sin\beta=-\frac{3}{4}$

$\therefore\ \cos(\alpha+\beta)=-\frac{3}{4}$ $\boxed{\cos\alpha\cos\beta-\sin\alpha\sin\beta=\cos(\alpha+\beta)}$

. Find the value of $\sin(\alpha+\beta)$ when $\sin\alpha+\cos\beta=\sin\gamma$ …① and $\cos\alpha+\sin\beta=\cos\gamma$ …②.

ol] From ①, $\sin^2\alpha+2\sin\alpha\cos\beta+\cos^2\beta=\sin^2\gamma$ …③

From ②, $\cos^2\alpha+2\cos\alpha\sin\beta+\sin^2\beta=\cos^2\gamma$ …④

From ③+④, $2+2(\sin\alpha\cos\beta+\cos\alpha\sin\beta)=1$

$\therefore\ \sin\alpha\cos\beta+\cos\alpha\sin\beta=-\frac{1}{2}$

$\therefore\ \sin(\alpha+\beta)=-\frac{1}{2}$

Addition Formulas 1

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Time : to :

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Ex.

Prove the identity $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

[Sol] $\tan(\alpha+\beta)=\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$ ← $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$

$=\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$ ← $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$, $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$=\dfrac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{\sin\beta}{\cos\beta}}$ ← Dividing the numerator and the denominator by $\cos\alpha\cos\beta$

$=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ ← $\dfrac{\sin\theta}{\cos\theta}=\tan\theta$

1. Prove the identity $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$.

[Sol] $\tan(\alpha-\beta)=\dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$

$=\dfrac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}$ ← $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$, $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$=\dfrac{\dfrac{\sin\alpha}{\cos\alpha}-\dfrac{\sin\beta}{\cos\beta}}{1+\dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{\sin\beta}{\cos\beta}}$

$=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

M155b

Addition Formulas II

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

2. Evaluate the following expressions using the formulas above.

(1) $\tan 75^\circ=\tan(45^\circ+30^\circ)$

$$=\frac{\tan 45^\circ+\tan 30^\circ}{1-\tan 45^\circ\tan 30^\circ}$$

$$=\frac{1+\frac{\sqrt{3}}{3}}{1-1\cdot\frac{\sqrt{3}}{3}}$$

$$=\frac{3+\sqrt{3}}{3-\sqrt{3}}$$

$$=\frac{12+6\sqrt{3}}{6}$$

$$=2+\sqrt{3}$$

Rationalizing the denominator as

$$\frac{3+\sqrt{3}}{3-\sqrt{3}}=\frac{(3+\sqrt{3})^2}{(3-\sqrt{3})(3+\sqrt{3})}\ \text{(J75)}$$

(2) $\tan 15^\circ=\tan(45^\circ-30^\circ)$

$$=\frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ\tan 30^\circ}$$

$$=\frac{1-\frac{\sqrt{3}}{3}}{1+1\cdot\frac{\sqrt{3}}{3}}$$

$$=\frac{3-\sqrt{3}}{3+\sqrt{3}}$$

$$=\frac{12-6\sqrt{3}}{6}$$

$$=2-\sqrt{3}$$

Alternative Solution

$\tan 15^\circ=\tan(60^\circ-45^\circ)$

$$=\frac{\tan 60^\circ-\tan 45^\circ}{1+\tan 60^\circ\tan 45^\circ}$$

$$=\frac{\sqrt{3}-1}{1+\sqrt{3}\cdot 1}$$

$$=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$=\frac{4-2\sqrt{3}}{2}$$

$$=2-\sqrt{3}$$

M156a

KUMON

Addition Formulas 1

Name

Date / /

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EX. Let $0<\alpha<\frac{\pi}{2}$ and $0<\beta<\frac{\pi}{2}$. Find the value of $\alpha+\beta$ when $\tan\alpha=2$ and $\tan\beta=3$.

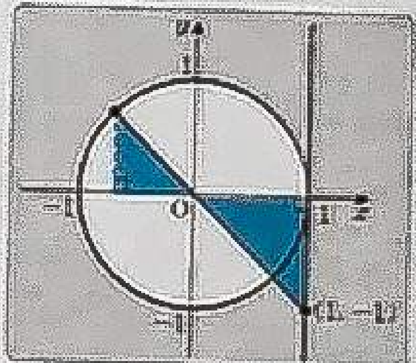
[Sol] $\tan(\alpha+\beta)=\frac{2+3}{1-2\cdot 3}$ ← $\boxed{\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}$

$=-1$

Since $0<\alpha<\frac{\pi}{2}$, $0<\beta<\frac{\pi}{2}$,

$0<\alpha+\beta<\pi$

$\therefore\ \alpha+\beta=\frac{3}{4}\pi$ ←



1. Let $0<\alpha<\frac{\pi}{2}$ and $0<\beta<\frac{\pi}{2}$. Find the value of $\alpha+\beta$ when $\tan\alpha=\frac{1}{2}$ and $\tan\beta=\frac{1}{3}$.

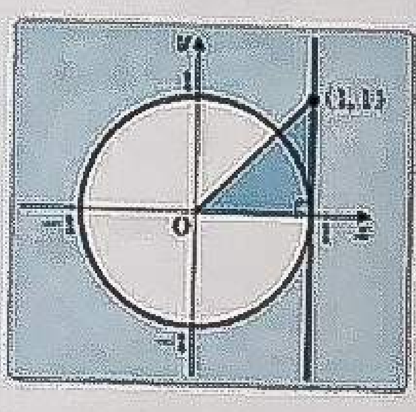
[Sol] $\tan(\alpha+\beta)=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}$

$=1$

Since $0<\alpha<\frac{\pi}{2}$, $0<\beta<\frac{\pi}{2}$,

$0<\alpha+\beta<\pi$

$\therefore\ \alpha+\beta=\frac{\pi}{4}$ ←



M156b

2. Let $0<\alpha<\frac{\pi}{2}$ and $0<\beta<\frac{\pi}{2}$. Find the value of $\alpha+\beta$ when $\tan\alpha=\frac{\sqrt{3}}{7}$ an
$\tan\beta=\frac{\sqrt{3}}{6}$.

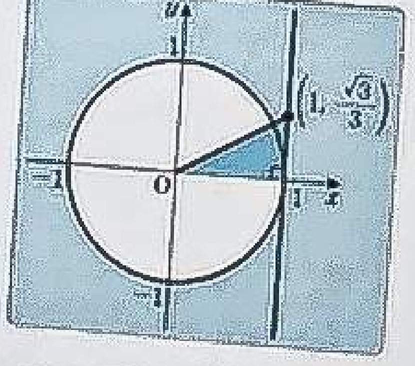
[Sol] $\tan(\alpha+\beta)=\dfrac{\frac{\sqrt{3}}{7}+\frac{\sqrt{3}}{6}}{1-\frac{\sqrt{3}}{7}\cdot\frac{\sqrt{3}}{6}}$

$=\frac{\sqrt{3}}{3}$

Since $0<\alpha<\frac{\pi}{2}$, $0<\beta<\frac{\pi}{2}$,

$0<\alpha+\beta<\pi$

$\therefore\ \alpha+\beta=\frac{\pi}{6}$



3. Let $0<\alpha<\frac{\pi}{2}$ and $-\frac{\pi}{2}<\beta<0$. Find the value of $\alpha-\beta$ when $\tan\alpha=2$ and $\tan\beta=-3$.

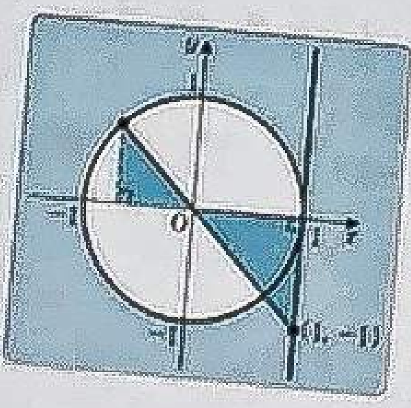
Sol] $\tan(\alpha-\beta)=\dfrac{2-(-3)}{1+2\cdot(-3)}$ $\Leftarrow$ $\boxed{\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}$

$=-1$

Since $0<\alpha<\frac{\pi}{2}$, $-\frac{\pi}{2}<\beta<0$,

$0<\alpha-\beta<\pi$

$\therefore\ \alpha-\beta=\frac{3}{4}\pi$



Addition Formulas 1

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Ex.

Let $\frac{\pi}{2}<\alpha<\pi$ and $0<\beta<\frac{\pi}{2}$. Find the value of $\tan(\alpha+\beta)$ when $\sin\alpha=\frac{3}{5}$ and $\cos\beta=\frac{12}{13}$.

$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$

[Sol] Since $\frac{\pi}{2}<\alpha<\pi$, $\cos\alpha<0$

$$\cos\alpha=-\sqrt{1-\left(\frac{3}{5}\right)^2}=-\frac{4}{5} \quad \therefore\ \tan\alpha=\frac{3}{5}\div\left(-\frac{4}{5}\right)=-\frac{3}{4}$$

Also, since $0<\beta<\frac{\pi}{2}$, $\sin\beta>0$

$$\sin\beta=\sqrt{1-\left(\frac{12}{13}\right)^2}=\frac{5}{13} \quad \therefore\ \tan\beta=\frac{5}{13}\div\frac{12}{13}=\frac{5}{12}$$

$$\therefore\ \tan(\alpha+\beta)=\frac{-\frac{3}{4}+\frac{5}{12}}{1-\left(-\frac{3}{4}\right)\cdot\frac{5}{12}}=-\frac{16}{63} \longleftarrow \tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

1. Let $0<\alpha<\frac{\pi}{2}$ and $\frac{\pi}{2}<\beta<\pi$. Find the value of $\tan(\alpha+\beta)$ when $\sin\alpha=\frac{4}{5}$ and $\cos\beta=-\frac{8}{17}$.

[Sol] Since $0<\alpha<\frac{\pi}{2}$, $\cos\alpha>0$

$$\cos\alpha=\sqrt{1-\left(\frac{4}{5}\right)^2}=\frac{3}{5} \quad \therefore\ \tan\alpha=\frac{4}{5}\div\frac{3}{5}=\frac{4}{3}$$

Also, since $\frac{\pi}{2}<\beta<\pi$, $\sin\beta>0$

$$\sin\beta=\sqrt{1-\left(-\frac{8}{17}\right)^2}=\frac{15}{17} \quad \therefore\ \tan\beta=\frac{15}{17}\div\left(-\frac{8}{17}\right)=-\frac{15}{8}$$

$$\therefore\ \tan(\alpha+\beta)=\frac{\frac{4}{3}+\left(-\frac{15}{8}\right)}{1-\frac{4}{3}\cdot\left(-\frac{15}{8}\right)}=-\frac{13}{84}$$

M157b

2. Let $0<\alpha<\frac{\pi}{2}$ and $-\frac{\pi}{2}<\beta<0$. Find the value of $\tan(\alpha+\beta)$ when $\sin\alpha=\frac{1}{3}$ and $\cos\beta=\frac{1}{3}$.

[Sol] Since $0<\alpha<\frac{\pi}{2}$, $\cos\alpha>0$

$$\cos\alpha=\sqrt{1-\left(\frac{1}{3}\right)^2}=\frac{2\sqrt{2}}{3} \quad \therefore\ \tan\alpha=\frac{1}{3}\div\frac{2\sqrt{2}}{3}=\frac{\sqrt{2}}{4}$$

Also, since $-\frac{\pi}{2}<\beta<0$, $\sin\beta<0$

$$\sin\beta=-\sqrt{1-\left(\frac{1}{3}\right)^2}=-\frac{2\sqrt{2}}{3} \quad \therefore\ \tan\beta=-\frac{2\sqrt{2}}{3}\div\frac{1}{3}=-2\sqrt{2}$$

$$\therefore\ \tan(\alpha+\beta)=\frac{\frac{\sqrt{2}}{4}+(-2\sqrt{2})}{1-\frac{\sqrt{2}}{4}\cdot(-2\sqrt{2})}=-\frac{7\sqrt{2}}{8}$$

3. Let $0<\alpha<\frac{\pi}{2}$ and $\frac{\pi}{2}<\beta<\pi$. Find the value of $\tan(\alpha-\beta)$ when $\sin\alpha=\frac{1}{2}$ and $\sin\beta=\frac{1}{3}$.

[Sol] Since $0<\alpha<\frac{\pi}{2}$, $\cos\alpha>0$

$$\cos\alpha=\sqrt{1-\left(\frac{1}{2}\right)^2}=\frac{\sqrt{3}}{2} \quad \therefore\ \tan\alpha=\frac{1}{2}\div\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{3}$$

Also, since $\frac{\pi}{2}<\beta<\pi$, $\cos\beta<0$

$$\cos\beta=-\sqrt{1-\left(\frac{1}{3}\right)^2}=-\frac{2\sqrt{2}}{3} \quad \therefore\ \tan\beta=\frac{1}{3}\div\left(-\frac{2\sqrt{2}}{3}\right)=-\frac{\sqrt{2}}{4}$$

$$\therefore\ \tan(\alpha-\beta)=\frac{\frac{\sqrt{3}}{3}-\left(-\frac{\sqrt{2}}{4}\right)}{1+\frac{\sqrt{3}}{3}\cdot\left(-\frac{\sqrt{2}}{4}\right)}$$

$$=\frac{4\sqrt{3}+3\sqrt{2}}{12-\sqrt{6}}$$

$$=\frac{9\sqrt{3}+8\sqrt{2}}{23}$$

$$\frac{4\sqrt{3}+3\sqrt{2}}{12-\sqrt{6}}=\frac{(4\sqrt{3}+3\sqrt{2})(12+\sqrt{6})}{(12-\sqrt{6})(12+\sqrt{6})}$$

M158a

KUMON

Addition Formulas 1

Name

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Time : to :

100%	~90%	~80%	~70%	69%~

Ex. Find the angle θ formed by two lines $y=3x-6$ and $y=\frac{1}{2}x+2$. $\left(0<\theta<\frac{\pi}{2}\right)$

[Sol] Let α and β be the angles formed by the positive x-axis and two corresponding parallel lines $y=3x$ and $y=\frac{1}{2}x$ which pass through the origin.

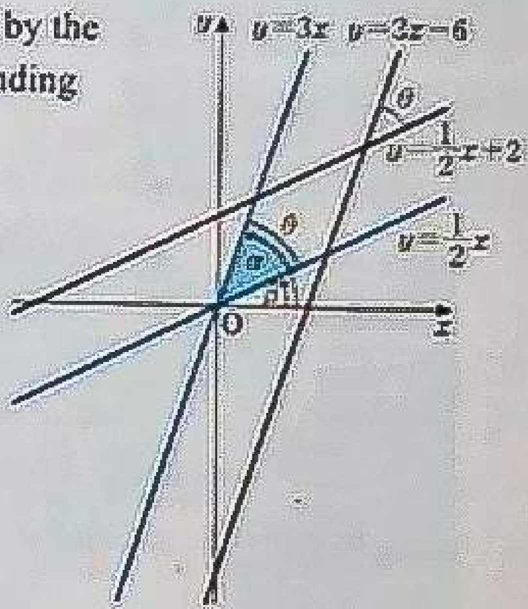
$\tan\alpha=3$, $\tan\beta=\frac{1}{2}$

Since $\theta=\alpha-\beta$,

$$\tan\theta=\tan(\alpha-\beta)$$

$$=\frac{3-\frac{1}{2}}{1+3\cdot\frac{1}{2}}=1$$

Since $0<\theta<\frac{\pi}{2}$, $\theta=\frac{\pi}{4}$



1 Find the angle θ formed by two lines $y=-2x+4$ and $y=3x-3$. $\left(0<\theta<\frac{\pi}{2}\right)$

[Sol] Let α and β be the angles formed by the positive x-axis and two corresponding parallel lines $y=-2x$ and $y=3x$ which pass through the origin.

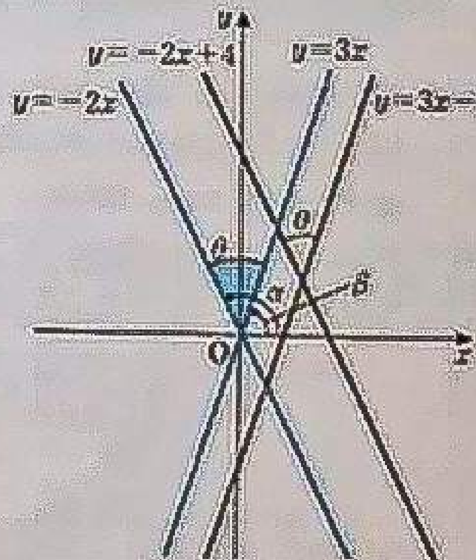
$\tan\alpha=-2$, $\tan\beta=3$

Since $\theta=\alpha-\beta$,

$$\tan\theta=\tan(\alpha-\beta)$$

$$=\frac{-2-3}{1+(-2)\cdot 3}=1$$

Since $0<\theta<\frac{\pi}{2}$, $\theta=\frac{\pi}{4}$



M158b

2. Find the angle θ formed by two lines $\sqrt{3}x-2y+4=0$ and $3\sqrt{3}x+y+5=0$. $\left(0<\theta<\frac{\pi}{2}\right)$

[Sol] Let α and β be the angles formed by the positive x-axis and two corresponding parallel lines $y=\frac{\sqrt{3}}{2}x$ and $y=-3\sqrt{3}x$ which pass through the origin.

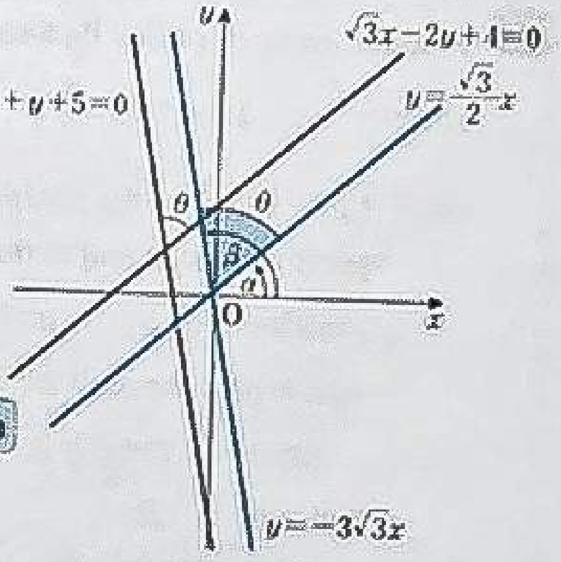
$\tan\alpha=\frac{\sqrt{3}}{2}$, $\tan\beta=-3\sqrt{3}$

Since $\theta=\beta-\alpha$, ⟵ From the diagram

$\tan\theta=\tan(\beta-\alpha)$

$$=\frac{-3\sqrt{3}-\frac{\sqrt{3}}{2}}{1+(-3\sqrt{3})\cdot\frac{\sqrt{3}}{2}}=\sqrt{3}$$

Since $0<\theta<\frac{\pi}{2}$, $\theta=\frac{\pi}{3}$



3. Find the value of m when the angle formed by line $y=mx(m>0)$ and the x-axis is equal to the angle θ formed by two lines $y=2x$ and $y=3x$. $\left(0<\theta<\frac{\pi}{2}\right)$

[Sol] Let α and β be the angles formed by the positive x-axis and two lines $y=2x$ and $y=3x$ respectively.

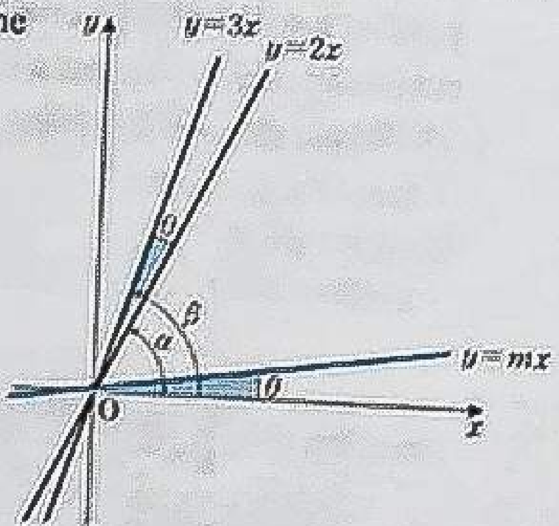
$\tan\alpha=2$, $\tan\beta=3$

Since $\theta=\beta-\alpha$,

$\tan\theta=\tan(\beta-\alpha)$

$$=\frac{3-2}{1+3\cdot2}=\frac{1}{7}$$

$\therefore\ m=\frac{1}{7}$



Addition Formulas 1

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Name

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Time : to :

1. Given $\tan\alpha=2$, $\tan\beta=5$, $\tan\gamma=8$ and $0<\alpha<\frac{\pi}{2}$, $0<\beta<\frac{\pi}{2}$, $0<\gamma<\frac{\pi}{2}$, find the values of $\tan(\alpha+\beta+\gamma)$ and $\alpha+\beta+\gamma$.

[Sol] $\tan(\alpha+\beta)=\dfrac{2+5}{1-2\cdot5}=-\dfrac{7}{9}$ ← $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

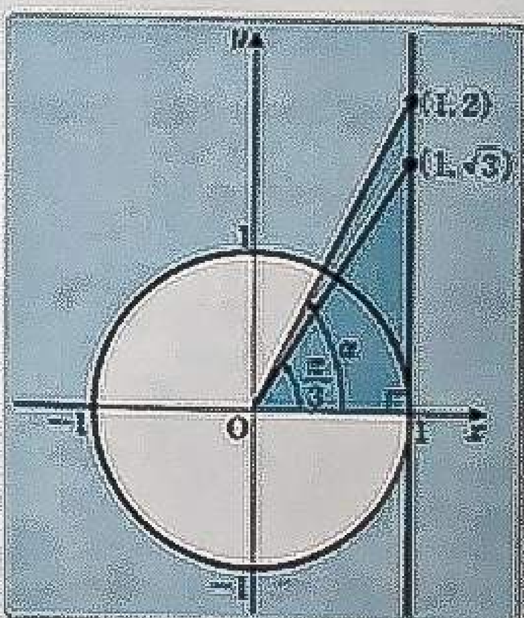
$$\tan(\alpha+\beta+\gamma)=\frac{-\frac{7}{9}+8}{1-\left(-\frac{7}{9}\right)\cdot8}=1 \quad \cdots①$$

← $\tan(\alpha+\beta+\gamma)=\tan\{(\alpha+\beta)+\gamma\}$ $=\dfrac{\tan(\alpha+\beta)+\tan\gamma}{1-\tan(\alpha+\beta)\tan\gamma}$

Also, $\tan\alpha=2>\sqrt{3}=\tan\frac{\pi}{3}$

Since $0<\alpha<\frac{\pi}{2}$,

$\frac{\pi}{3}<\alpha<\frac{\pi}{2}$ $\cdots②$ ←



Similarly,

$\tan\beta=5>\sqrt{3}=\tan\frac{\pi}{3}$

Since $0<\beta<\frac{\pi}{2}$,

$\frac{\pi}{3}<\beta<\frac{\pi}{2}$ $\cdots③$

$\tan\gamma=8>\sqrt{3}=\tan\frac{\pi}{3}$

Since $0<\gamma<\frac{\pi}{2}$,

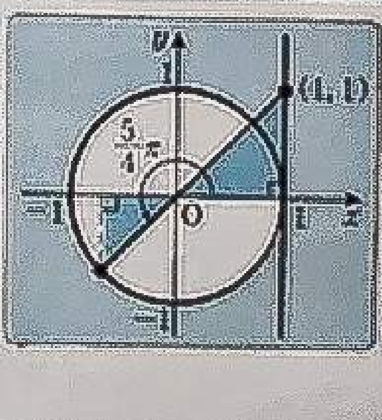
$\frac{\pi}{3}<\gamma<\frac{\pi}{2}$ $\cdots④$

From ②~④,

$\pi<\alpha+\beta+\gamma<\frac{3}{2}\pi$ $\cdots⑤$ ← Since $\frac{\pi}{3}+\frac{\pi}{3}+\frac{\pi}{3}<\alpha+\beta+\gamma<\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}$

From ① and ⑤,

$\alpha+\beta+\gamma=\frac{5}{4}\pi$ ←



M159b

2. Find all the values of $\alpha+\beta+\gamma$ when $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$.
$\left(-\frac{\pi}{2}<\alpha<\frac{\pi}{2},\ -\frac{\pi}{2}<\beta<\frac{\pi}{2},\ -\frac{\pi}{2}<\gamma<\frac{\pi}{2}\right)$

[Sol] $\tan(\alpha+\beta+\gamma)=\tan[(\alpha+\beta)+\gamma]$

$$=\frac{\tan(\alpha+\beta)+\tan\gamma}{1-\tan(\alpha+\beta)\tan\gamma}$$

$$=\frac{\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}+\tan\gamma}{1-\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\cdot\tan\gamma}$$

Multiplying the numerator and the denominator by $1-\tan\alpha\tan\beta$

$$=\frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\beta\tan\gamma-\tan\gamma\tan\alpha}\ \cdots①$$

Substituting $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$ into ①,

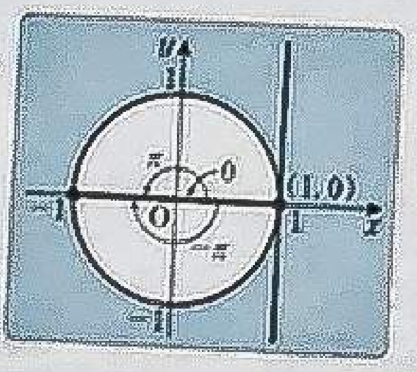
$\tan(\alpha+\beta+\gamma)=0\ \cdots②$

Since $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, $-\frac{\pi}{2}<\beta<\frac{\pi}{2}$ and $-\frac{\pi}{2}<\gamma<\frac{\pi}{2}$,

$-\frac{3}{2}\pi<\alpha+\beta+\gamma<\frac{3}{2}\pi\ \cdots③$

From ② and ③,

$\alpha+\beta+\gamma=-\pi, 0, \pi$



Addition Formulas 1

Name

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Time : to :

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1. Let $0<\alpha<\frac{\pi}{2}$ and $\frac{\pi}{2}<\beta<\pi$. Find the value of $\sin(\alpha+\beta)$ when $\sin\alpha=\frac{4}{5}$ and $\sin\beta=\frac{12}{13}$.

➡ MI53

[Sol] Since $0<\alpha<\frac{\pi}{2}$, $\cos\alpha>0$

$$\cos\alpha=\sqrt{1-\left(\frac{4}{5}\right)^2}=\frac{3}{5}$$

Also, since $\frac{\pi}{2}<\beta<\pi$, $\cos\beta<0$

$$\cos\beta=-\sqrt{1-\left(\frac{12}{13}\right)^2}=-\frac{5}{13}$$

Therefore,

$$\sin(\alpha+\beta)=\frac{4}{5}\cdot\left(-\frac{5}{13}\right)+\frac{3}{5}\cdot\frac{12}{13}$$
$$=\frac{16}{65}$$

2. Let $\frac{\pi}{2}<\alpha<\pi$ and $\pi<\beta<\frac{3}{2}\pi$. Find the value of $\cos(\alpha-\beta)$ when $\sin\alpha=\frac{\sqrt{3}}{2}$ and $\cos\beta=-\frac{3}{5}$.

➡ MI53

[Sol] Since $\frac{\pi}{2}<\alpha<\pi$, $\cos\alpha<0$

$$\cos\alpha=-\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}=-\frac{1}{2}$$

Also, since $\pi<\beta<\frac{3}{2}\pi$, $\sin\beta<0$

$$\sin\beta=-\sqrt{1-\left(-\frac{3}{5}\right)^2}=-\frac{4}{5}$$

Therefore,

$$\cos(\alpha-\beta)=-\frac{1}{2}\cdot\left(-\frac{3}{5}\right)+\frac{\sqrt{3}}{2}\cdot\left(-\frac{4}{5}\right)$$
$$=\frac{3-4\sqrt{3}}{10}$$

M160b

3. Let $0 < \alpha < \frac{\pi}{2}$ and $\pi < \beta < \frac{3}{2}\pi$. Find the value of $\tan(\alpha+\beta)$ when $\sin\alpha = \frac{5}{13}$ and $\cos\beta = -\frac{12}{13}$.

➡ M157

[Sol] Since $0 < \alpha < \frac{\pi}{2}$, $\cos\alpha > 0$

$$\cos\alpha = \sqrt{1-\left(\frac{5}{13}\right)^2} = \frac{12}{13} \quad \therefore \tan\alpha = \frac{5}{13} \div \frac{12}{13} = \frac{5}{12}$$

Also, since $\pi < \beta < \frac{3}{2}\pi$, $\sin\beta < 0$

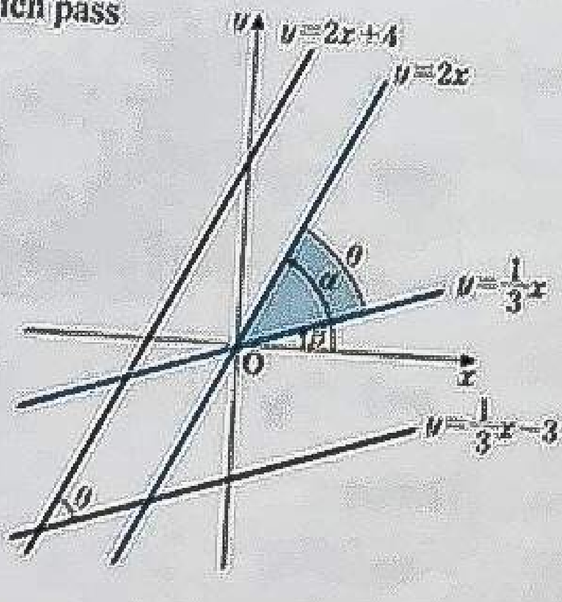
$$\sin\beta = -\sqrt{1-\left(-\frac{12}{13}\right)^2} = -\frac{5}{13} \quad \therefore \tan\beta = -\frac{5}{13} \div \left(-\frac{12}{13}\right) = \frac{5}{12}$$

$$\therefore \tan(\alpha+\beta) = \frac{\frac{5}{12}+\frac{5}{12}}{1-\frac{5}{12}\cdot\frac{5}{12}} = \frac{120}{119}$$

4. Find the angle θ formed by two lines $y=2x+4$ and $y=\frac{1}{3}x-3$. $\left(0<\theta<\frac{\pi}{2}\right)$

[Sol] Let α and β be the angles formed by the positive x-axis and two corresponding parallel lines $y=2x$ and $y=\frac{1}{3}x$ which pass through the origin.

➡ M158



$\tan\alpha = 2$, $\tan\beta = \frac{1}{3}$

Since $\theta = \alpha - \beta$,

$$\tan\theta = \tan(\alpha-\beta) = \frac{2-\frac{1}{3}}{1+2\cdot\frac{1}{3}} = 1$$

Since $0<\theta<\frac{\pi}{2}$, $\theta = \frac{\pi}{4}$

Addition Formulas 2

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Name

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1. Fill in the following blanks.

(1) Express $\sin 2\alpha$ in terms of $\sin\alpha$ and $\cos\alpha$ by substituting α for β on both sides of identity $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$.

[Sol] $\sin(\alpha+\alpha)=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha$

$\therefore\ \sin 2\alpha=\boxed{2\sin\alpha\cos\alpha}$

(2) Express $\cos 2\alpha$ in terms of $\sin\alpha$ and $\cos\alpha$ by substituting α for β on both sides of identity $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$.

[Sol] $\cos(\alpha+\alpha)=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha$

$\therefore\ \cos 2\alpha=\boxed{\cos^2\alpha-\sin^2\alpha}$ $\cdots$①

Also, since $\sin^2\alpha+\cos^2\alpha=1$,

substituting $\cos^2\alpha=1-\sin^2\alpha$ into ①,

$\cos 2\alpha=\boxed{1-2\sin^2\alpha}$

Substituting $\sin^2\alpha=1-\cos^2\alpha$ into ①,

$\cos 2\alpha=\boxed{2\cos^2\alpha-1}$

(3) Express $\tan 2\alpha$ in terms of $\tan\alpha$ by substituting α for β on both sides of identity $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

[Sol] $\tan(\alpha+\alpha)=\dfrac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}$

$\therefore\ \tan 2\alpha=\boxed{\dfrac{2\tan\alpha}{1-\tan^2\alpha}}$

M161b

Double-Angle Formulas

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$= 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$

$\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

2. Solve the following questions using the formulas above.

(1) Find the value of $\sin 2\alpha$ when $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$.

[Sol] $\sin 2\alpha = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$

(2) Find the value of $\cos 2\alpha$ when $\sin\alpha = -\frac{5}{13}$ and $\cos\alpha = \frac{12}{13}$.

[Sol] $\cos 2\alpha = \left(\frac{12}{13}\right)^2 - \left(-\frac{5}{13}\right)^2 = \frac{119}{169}$

[Alternative Solution 1
$\cos 2\alpha = 1 - 2\cdot\left(-\frac{5}{13}\right)^2 = \frac{119}{169}$]
[Alternative Solution 2
$\cos 2\alpha = 2\cdot\left(\frac{12}{13}\right)^2 - 1 = \frac{119}{169}$]

3) Find the value of $\cos 2\alpha$ when $\sin\alpha = \frac{\sqrt{5}}{3}$.

Sol] $\cos 2\alpha = 1 - 2\cdot\left(\frac{\sqrt{5}}{3}\right)^2 = -\frac{1}{9}$

) Find the value of $\tan 2\alpha$ when $\tan\alpha = -\frac{1}{3}$.

ol] $\tan 2\alpha = \dfrac{2\cdot\left(-\frac{1}{3}\right)}{1-\left(-\frac{1}{3}\right)^2} = -\frac{3}{4}$

Addition Formulas 2

Name

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Ex. Let $\frac{\pi}{2}<\alpha<\pi$. Find the values of $\cos 2\alpha$ and $\sin 2\alpha$ when $\sin\alpha=\frac{3}{5}$.

[Sol] $\cos 2\alpha=1-2\cdot\left(\frac{3}{5}\right)^2=\frac{7}{25}$ ← $\boxed{\cos 2\alpha=1-2\sin^2\alpha}$

Also, since $\frac{\pi}{2}<\alpha<\pi$, $\cos\alpha<0$

$$\cos\alpha=-\sqrt{1-\left(\frac{3}{5}\right)^2}=-\frac{4}{5}$$

$$\therefore\ \sin 2\alpha=2\cdot\frac{3}{5}\cdot\left(-\frac{4}{5}\right)=-\frac{24}{25}$$ ← $\boxed{\sin 2\alpha=2\sin\alpha\cos\alpha}$

1. Let $\frac{\pi}{2}<\alpha<\pi$. Find the values of $\cos 2\alpha$ and $\sin 2\alpha$ when $\cos\alpha=-\frac{5}{13}$.

[Sol] $\cos 2\alpha=2\cdot\left(-\frac{5}{13}\right)^2-1=-\frac{119}{169}$ ← $\boxed{\cos 2\alpha=2\cos^2\alpha-1}$

Also, since $\frac{\pi}{2}<\alpha<\pi$, $\sin\alpha>0$

$$\sin\alpha=\sqrt{1-\left(-\frac{5}{13}\right)^2}=\frac{12}{13}$$

$$\therefore\ \sin 2\alpha=2\cdot\frac{12}{13}\cdot\left(-\frac{5}{13}\right)=-\frac{120}{169}$$

2. Let $\pi<\alpha<\frac{3}{2}\pi$. Find the values of $\cos 2\alpha$ and $\sin 2\alpha$ when $\sin\alpha=-\frac{\sqrt{5}}{3}$.

[Sol] $\cos 2\alpha=1-2\cdot\left(-\frac{\sqrt{5}}{3}\right)^2=-\frac{1}{9}$

Also, since $\pi<\alpha<\frac{3}{2}\pi$, $\cos\alpha<0$

$$\cos\alpha=-\sqrt{1-\left(-\frac{\sqrt{5}}{3}\right)^2}=-\frac{2}{3}$$

$$\therefore\ \sin 2\alpha=2\cdot\left(-\frac{\sqrt{5}}{3}\right)\cdot\left(-\frac{2}{3}\right)=\frac{4\sqrt{5}}{9}$$

3. Let $\frac{3}{2}\pi < \alpha < 2\pi$. Find the values of $\cos 2\alpha$ and $\sin 2\alpha$ when $\cos\alpha = \frac{\sqrt{3}}{3}$.

[Sol] $\cos 2\alpha = 2\cdot\left(\frac{\sqrt{3}}{3}\right)^2 - 1 = -\frac{1}{3}$

Also, since $\frac{3}{2}\pi < \alpha < 2\pi$, $\sin\alpha < 0$

$$\sin\alpha = -\sqrt{1-\left(\frac{\sqrt{3}}{3}\right)^2} = -\frac{\sqrt{6}}{3}$$

$$\therefore\ \sin 2\alpha = 2\cdot\left(-\frac{\sqrt{6}}{3}\right)\cdot\frac{\sqrt{3}}{3} = -\frac{2\sqrt{2}}{3}$$

4. Let $\frac{\pi}{2} < \alpha < \pi$. Find the values of $\cos 2\alpha$ and $\sin 2\alpha$ when $\sin\alpha = \frac{2}{5}$.

[Sol] $\cos 2\alpha = 1 - 2\cdot\left(\frac{2}{5}\right)^2 = \frac{17}{25}$

Also, since $\frac{\pi}{2} < \alpha < \pi$, $\cos\alpha < 0$

$$\cos\alpha = -\sqrt{1-\left(\frac{2}{5}\right)^2} = -\frac{\sqrt{21}}{5}$$

$$\therefore\ \sin 2\alpha = 2\cdot\frac{2}{5}\cdot\left(-\frac{\sqrt{21}}{5}\right) = -\frac{4\sqrt{21}}{25}$$

5. Let $\pi < \alpha < \frac{3}{2}\pi$. Find the value of $\tan 2\alpha$ when $\sin\alpha = -\frac{1}{3}$.

[Sol] Since $\pi < \alpha < \frac{3}{2}\pi$, $\cos\alpha < 0$

$$\cos\alpha = -\sqrt{1-\left(-\frac{1}{3}\right)^2} = -\frac{2\sqrt{2}}{3}$$

$$\therefore\ \tan\alpha = -\frac{1}{3} \div \left(-\frac{2\sqrt{2}}{3}\right) = \frac{\sqrt{2}}{4}$$

$\boxed{\tan\alpha = \frac{\sin\alpha}{\cos\alpha}}$

Therefore,

$$\tan 2\alpha = \frac{2\cdot\frac{\sqrt{2}}{4}}{1-\left(\frac{\sqrt{2}}{4}\right)^2} = \frac{4\sqrt{2}}{7}$$

$\boxed{\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}}$

Alternative Solution

$\cos 2\alpha = 1 - 2\cdot\left(-\frac{1}{3}\right)^2 = \frac{7}{9}$

Also, since $\pi < \alpha < \frac{3}{2}\pi$, $\cos\alpha < 0$

$\cos\alpha = -\sqrt{1-\left(-\frac{1}{3}\right)^2} = -\frac{2\sqrt{2}}{3}$

$\therefore\ \sin 2\alpha = 2\cdot\left(-\frac{1}{3}\right)\cdot\left(-\frac{2\sqrt{2}}{3}\right)$

$= \frac{4\sqrt{2}}{9}$

$\therefore\ \tan 2\alpha = \frac{4\sqrt{2}}{9} \div \frac{7}{9} = \frac{4\sqrt{2}}{7}$

Addition Formulas 2

Name

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1. Prove the following identities.

Ex. $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

[Sol] $\sin 3\alpha = \sin(2\alpha + \alpha)$

$= \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha$ ← $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= 2\sin\alpha\cos^2\alpha + (1 - 2\sin^2\alpha)\sin\alpha$ ← $\sin 2\alpha = 2\sin\alpha\cos\alpha$, $\cos 2\alpha = 1 - 2\sin^2\alpha$

$= 2\sin\alpha(1 - \sin^2\alpha) + \sin\alpha - 2\sin^3\alpha$ ← $\cos^2\alpha = 1 - \sin^2\alpha$

$= 3\sin\alpha - 4\sin^3\alpha$

(1) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

[Sol] $\cos 3\alpha = \cos(2\alpha + \alpha)$

$= \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha$ ← $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= (2\cos^2\alpha - 1)\cos\alpha - 2\sin^2\alpha\cos\alpha$ ← $\cos 2\alpha = 2\cos^2\alpha - 1$

$= 2\cos^3\alpha - \cos\alpha - 2(1 - \cos^2\alpha)\cos\alpha$ ← $\sin^2\alpha = 1 - \cos^2\alpha$

$= 4\cos^3\alpha - 3\cos\alpha$

M163b

Triple-Angle Formulas

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

2. Solve the following questions using the formulas above.

(1) Find the value of $\sin 3\alpha$ when $\sin\alpha = \frac{3}{5}$.

[Sol] $\sin 3\alpha = 3\cdot\frac{3}{5} - 4\cdot\left(\frac{3}{5}\right)^3$ ⟵ $\boxed{\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha}$

$= \frac{117}{125}$

(2) Find the value of $\cos 3\alpha$ when $\cos\alpha = \frac{4}{5}$.

[Sol] $\cos 3\alpha = 4\cdot\left(\frac{4}{5}\right)^3 - 3\cdot\frac{4}{5}$ ⟵ $\boxed{\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha}$

$= -\frac{44}{125}$

(3) Let $\frac{\pi}{2} < \alpha < \pi$. Find the value of $\sin 3\alpha$ when $\cos\alpha = -\frac{\sqrt{5}}{3}$.

[Sol] Since $\frac{\pi}{2} < \alpha < \pi$, $\sin\alpha > 0$

$$\sin\alpha = \sqrt{1 - \left(-\frac{\sqrt{5}}{3}\right)^2} = \frac{2}{3}$$

$$\therefore\ \sin 3\alpha = 3\cdot\frac{2}{3} - 4\cdot\left(\frac{2}{3}\right)^3$$

$$= \frac{22}{27}$$

(4) Let $\frac{3}{2}\pi < \alpha < 2\pi$. Find the value of $\cos 3\alpha$ when $\sin\alpha = -\frac{\sqrt{6}}{3}$.

[Sol] Since $\frac{3}{2}\pi < \alpha < 2\pi$, $\cos\alpha > 0$

$$\cos\alpha = \sqrt{1 - \left(-\frac{\sqrt{6}}{3}\right)^2} = \frac{\sqrt{3}}{3}$$

$$\therefore\ \cos 3\alpha = 4\cdot\left(\frac{\sqrt{3}}{3}\right)^3 - 3\cdot\frac{\sqrt{3}}{3}$$

$$= -\frac{5\sqrt{3}}{9}$$

Addition Formulas 2

Name

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1. Fill in the following blanks.

(1) Express $\sin^2\frac{\alpha}{2}$ in terms of $\cos\alpha$ by substituting $\frac{\alpha}{2}$ for α on both sides of identity $\cos 2\alpha = 1 - 2\sin^2\alpha$.

[Sol] $\cos\left(2\cdot\frac{\alpha}{2}\right) = 1 - 2\sin^2\frac{\alpha}{2}$

$$\therefore \sin^2\frac{\alpha}{2} = \boxed{\frac{1-\cos\alpha}{2}}$$

(2) Express $\cos^2\frac{\alpha}{2}$ in terms of $\cos\alpha$ by substituting $\frac{\alpha}{2}$ for α on both sides of identity $\cos 2\alpha = 2\cos^2\alpha - 1$.

[Sol] $\cos\left(2\cdot\frac{\alpha}{2}\right) = 2\cos^2\frac{\alpha}{2} - 1$

$$\therefore \cos^2\frac{\alpha}{2} = \boxed{\frac{1+\cos\alpha}{2}}$$

(3) Express $\tan^2\frac{\alpha}{2}$ in terms of $\cos\alpha$ by substituting $\frac{\alpha}{2}$ for α on both sides of identity $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ and using the solutions of (1) and (2) above.

[Sol] Since $\tan\frac{\alpha}{2} = \dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$, $\tan^2\frac{\alpha}{2} = \dfrac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}$

From (1) and (2),

$$\tan^2\frac{\alpha}{2} = \frac{\boxed{\frac{1-\cos\alpha}{2}}}{\boxed{\frac{1+\cos\alpha}{2}}} = \boxed{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Half-Angle Formulas

$$\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2},\quad \cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2},\quad \tan^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{1+\cos\alpha}$$

2. Evaluate the following expressions using the formulas above.

EX. $\sin 22.5^\circ$

[Sol] $\sin^2 22.5^\circ=\frac{1-\cos 45^\circ}{2}=\frac{2-\sqrt{2}}{4}$ ⟵ $\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}$

Since $\sin 22.5^\circ>0$,

$$\sin 22.5^\circ=\frac{\sqrt{2-\sqrt{2}}}{2}$$

(1) $\cos 22.5^\circ$

[Sol] $\cos^2 22.5^\circ=\frac{1+\cos 45^\circ}{2}=\frac{2+\sqrt{2}}{4}$ ⟵ $\cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}$

Since $\cos 22.5^\circ>0$,

$$\cos 22.5^\circ=\frac{\sqrt{2+\sqrt{2}}}{2}$$

(2) $\tan 22.5^\circ$

[Sol] $\tan^2 22.5^\circ=\frac{1-\cos 45^\circ}{1+\cos 45^\circ}=\frac{2-\sqrt{2}}{2+\sqrt{2}}$ ⟵ $\tan^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{1+\cos\alpha}$

$=3-2\sqrt{2}$ ⟵ $\frac{2-\sqrt{2}}{2+\sqrt{2}}=\frac{(2-\sqrt{2})^2}{(2+\sqrt{2})(2-\sqrt{2})}$

Since $\tan 22.5^\circ>0$,

$\tan 22.5^\circ=\sqrt{3-2\sqrt{2}}=\sqrt{2}-1$ ⟵ $\sqrt{3-2\sqrt{2}}=\sqrt{(\sqrt{2}-1)^2}$

Addition Formulas 2

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Ex.

Let $\frac{\pi}{2}<\alpha<\pi$. Find the value of $\sin\frac{\alpha}{2}$ when $\sin\alpha=\frac{4}{5}$.

[Sol] Since $\frac{\pi}{2}<\alpha<\pi$, $\cos\alpha<0$

$$\cos\alpha=-\sqrt{1-\left(\frac{4}{5}\right)^2}=-\frac{3}{5}$$

$$\therefore\ \sin^2\frac{\alpha}{2}=\frac{1-\left(-\frac{3}{5}\right)}{2}=\frac{4}{5} \quad \longleftarrow \quad \boxed{\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}}$$

Since $\frac{\pi}{2}<\alpha<\pi$, $\frac{\pi}{4}<\frac{\alpha}{2}<\frac{\pi}{2}$; therefore, $\sin\frac{\alpha}{2}>0$

$$\therefore\ \sin\frac{\alpha}{2}=\frac{2\sqrt{5}}{5}$$

1. Let $\pi<\alpha<\frac{3}{2}\pi$. Find the value of $\cos\frac{\alpha}{2}$ when $\sin\alpha=-\frac{3}{5}$.

[Sol] Since $\pi<\alpha<\frac{3}{2}\pi$, $\cos\alpha<0$

$$\cos\alpha=-\sqrt{1-\left(-\frac{3}{5}\right)^2}=-\frac{4}{5}$$

$$\therefore\ \cos^2\frac{\alpha}{2}=\frac{1+\left(-\frac{4}{5}\right)}{2}=\frac{1}{10} \quad \longleftarrow \quad \boxed{\cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}}$$

Since $\pi<\alpha<\frac{3}{2}\pi$, $\frac{\pi}{2}<\frac{\alpha}{2}<\frac{3}{4}\pi$; therefore, $\cos\frac{\alpha}{2}<0$

$$\therefore\ \cos\frac{\alpha}{2}=-\frac{\sqrt{10}}{10}$$

2. Let $0<\alpha<\frac{\pi}{2}$. Find the value of $\sin\frac{\alpha}{2}$ when $\cos\alpha=\frac{3}{5}$.

[Sol] $\sin^2\frac{\alpha}{2}=\frac{1-\frac{3}{5}}{2}=\frac{1}{5}$

Since $0<\alpha<\frac{\pi}{2}$, $0<\frac{\alpha}{2}<\frac{\pi}{4}$; therefore, $\sin\frac{\alpha}{2}>0$

$$\therefore\ \sin\frac{\alpha}{2}=\frac{\sqrt{5}}{5}$$

3. Let $\pi < \alpha < \frac{3}{2}\pi$. Find the value of $\sin\frac{\alpha}{2}$ when $\sin\alpha = -\frac{2\sqrt{2}}{3}$.

[Sol] Since $\pi < \alpha < \frac{3}{2}\pi$, $\cos\alpha < 0$

$$\cos\alpha = -\sqrt{1-\left(-\frac{2\sqrt{2}}{3}\right)^2} = -\frac{1}{3}$$

$$\therefore\ \sin^2\frac{\alpha}{2} = \frac{1-\left(-\frac{1}{3}\right)}{2} = \frac{2}{3}$$

Since $\pi < \alpha < \frac{3}{2}\pi$, $\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3}{4}\pi$; therefore, $\sin\frac{\alpha}{2} > 0$

$$\therefore\ \sin\frac{\alpha}{2} = \frac{\sqrt{6}}{3}$$

4. Let $\frac{3}{2}\pi < \alpha < 2\pi$. Find the value of $\cos\frac{\alpha}{2}$ when $\cos\alpha = \frac{5}{13}$.

[Sol] $\cos^2\frac{\alpha}{2} = \frac{1+\frac{5}{13}}{2} = \frac{9}{13}$

Since $\frac{3}{2}\pi < \alpha < 2\pi$, $\frac{3}{4}\pi < \frac{\alpha}{2} < \pi$; therefore, $\cos\frac{\alpha}{2} < 0$

$$\therefore\ \cos\frac{\alpha}{2} = -\frac{3\sqrt{13}}{13}$$

5. Let $\frac{\pi}{2} < \alpha < \pi$. Find the value of $\tan\frac{\alpha}{2}$ when $\sin\alpha = \frac{\sqrt{3}}{2}$.

[Sol] Since $\frac{\pi}{2} < \alpha < \pi$, $\cos\alpha < 0$

$$\cos\alpha = -\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2} = -\frac{1}{2}$$

$$\therefore\ \tan^2\frac{\alpha}{2} = \frac{1-\left(-\frac{1}{2}\right)}{1+\left(-\frac{1}{2}\right)} = 3 \quad \Longleftarrow \quad \boxed{\tan^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha}}$$

Since $\frac{\pi}{2} < \alpha < \pi$, $\frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2}$; therefore, $\tan\frac{\alpha}{2} > 0$

$$\therefore\ \tan\frac{\alpha}{2} = \sqrt{3}$$

M166a

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Given $0 \leqq \theta < 2\pi$, solve the following equations.

Ex. $\cos 2\theta + 3\cos\theta - 1 = 0$

[Sol] $(2\cos^2\theta - 1) + 3\cos\theta - 1 = 0$ ← $\cos 2\alpha = 2\cos^2\alpha - 1$

$2\cos^2\theta + 3\cos\theta - 2 = 0$

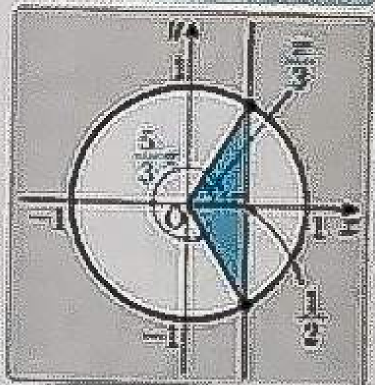
$(2\cos\theta - 1)(\cos\theta + 2) = 0$ ← $2t^2 + 3t - 2 = 0$, $(2t-1)(t+2) = 0$

$\cos\theta = \frac{1}{2},\ -2$

Since $-1 \leqq \cos\theta \leqq 1$,

$\cos\theta = \frac{1}{2}$

$\therefore\ \theta = \frac{\pi}{3},\ \frac{5}{3}\pi$ ←



(1) $5\sin\theta + \cos 2\theta - 3 = 0$

[Sol] $5\sin\theta + (1 - 2\sin^2\theta) - 3 = 0$ ← $\cos 2\alpha = 1 - 2\sin^2\alpha$

$2\sin^2\theta - 5\sin\theta + 2 = 0$

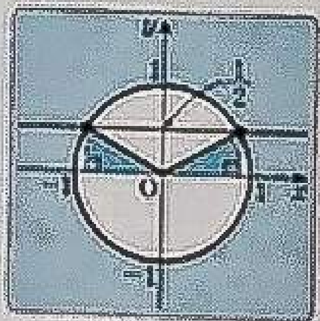
$(2\sin\theta - 1)(\sin\theta - 2) = 0$

$\sin\theta = \frac{1}{2},\ 2$

Since $-1 \leqq \sin\theta \leqq 1$,

$\sin\theta = \frac{1}{2}$

$\therefore\ \theta = \frac{\pi}{6},\ \frac{5}{6}\pi$ ←



(2) $\cos 2\theta - \cos\theta = 0$

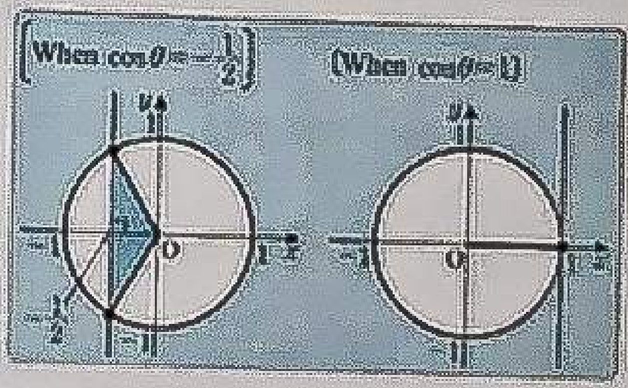
[Sol] $(2\cos^2\theta - 1) - \cos\theta = 0$

$2\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\cos\theta = -\frac{1}{2},\ 1$

$\therefore\ \theta = 0,\ \frac{2}{3}\pi,\ \frac{4}{3}\pi$ ←



M166b

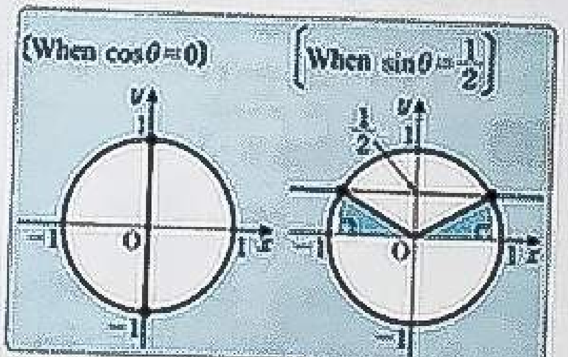
(3) $\sin 2\theta = \cos\theta$

[Sol] $2\sin\theta\cos\theta - \cos\theta = 0$ $\boxed{\sin 2\alpha = 2\sin\alpha\cos\alpha}$

$\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0,\ \sin\theta = \dfrac{1}{2}$

$\therefore\ \theta = \dfrac{\pi}{6},\ \dfrac{\pi}{2},\ \dfrac{5}{6}\pi,\ \dfrac{3}{2}\pi$



(4) $\sin\theta - \cos\dfrac{\theta}{2} = 0$

[Sol] $\sin\left(2\cdot\dfrac{\theta}{2}\right) - \cos\dfrac{\theta}{2} = 0$

$2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} - \cos\dfrac{\theta}{2} = 0$

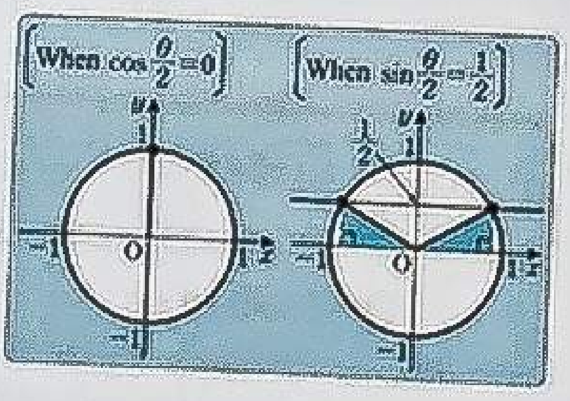
$\cos\dfrac{\theta}{2}\left(2\sin\dfrac{\theta}{2} - 1\right) = 0$

$\cos\dfrac{\theta}{2} = 0,\ \sin\dfrac{\theta}{2} = \dfrac{1}{2}$

Also, since $0 \leqq \theta < 2\pi,\ 0 \leqq \dfrac{\theta}{2} < \pi$

$\therefore\ \dfrac{\theta}{2} = \dfrac{\pi}{6},\ \dfrac{\pi}{2},\ \dfrac{5}{6}\pi$

$\therefore\ \theta = \dfrac{\pi}{3},\ \pi,\ \dfrac{5}{3}\pi$



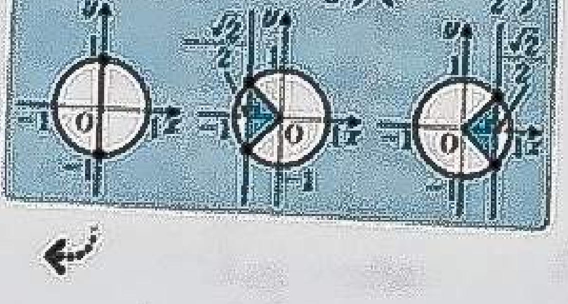
(5) $\cos 3\theta + \cos\theta = 0$

[Sol] $(4\cos^3\theta - 3\cos\theta) + \cos\theta = 0$ $\boxed{\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha}$

$4\cos^3\theta - 2\cos\theta = 0$

$2\cos\theta(\sqrt{2}\cos\theta + 1)(\sqrt{2}\cos\theta - 1) = 0$

$\cos\theta = 0,\ -\dfrac{\sqrt{2}}{2},\ \dfrac{\sqrt{2}}{2}$



$\therefore\ \theta = \dfrac{\pi}{4},\ \dfrac{\pi}{2},\ \dfrac{3}{4}\pi,\ \dfrac{5}{4}\pi,\ \dfrac{3}{2}\pi,\ \dfrac{7}{4}\pi$

Addition Formulas 2

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Given $0 \leqq \theta < 2\pi$, solve the following inequalities.

Ex. $\cos 2\theta - 5\sin\theta - 3 < 0$

[Sol] $(1-2\sin^2\theta) - 5\sin\theta - 3 < 0$

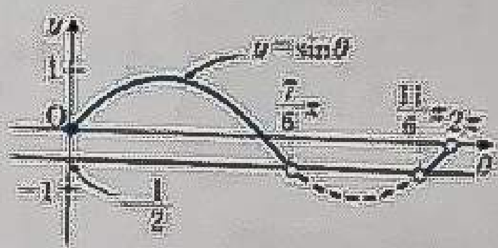
$2\sin^2\theta + 5\sin\theta + 2 > 0$

$(2\sin\theta + 1)(\sin\theta + 2) > 0$

Since $-1 \leqq \sin\theta \leqq 1$, always $\sin\theta + 2 > 0$.

$\therefore\ 2\sin\theta + 1 > 0$

$\sin\theta > -\dfrac{1}{2}$



When $\sin\theta = -\dfrac{1}{2}$, $\theta = \dfrac{7}{6}\pi,\ \dfrac{11}{6}\pi$

Therefore,

$0 \leqq \theta < \dfrac{7}{6}\pi,\ \dfrac{11}{6}\pi < \theta < 2\pi$

(1) $\cos 2\theta - 3\sqrt{3}\cos\theta + 4 > 0$

[Sol] $(2\cos^2\theta - 1) - 3\sqrt{3}\cos\theta + 4 > 0$

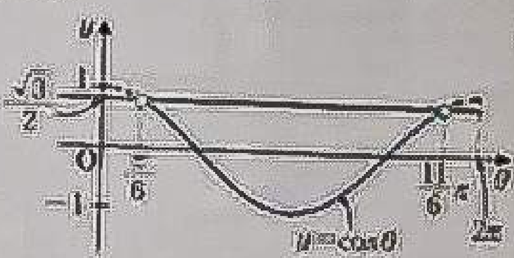
$2\cos^2\theta - 3\sqrt{3}\cos\theta + 3 > 0$

$(2\cos\theta - \sqrt{3})(\cos\theta - \sqrt{3}) > 0$

Since $-1 \leqq \cos\theta \leqq 1$, always $\cos\theta - \sqrt{3} < 0$.

$\therefore\ 2\cos\theta - \sqrt{3} < 0$

$\cos\theta < \dfrac{\sqrt{3}}{2}$



When $\cos\theta = \dfrac{\sqrt{3}}{2}$, $\theta = \dfrac{\pi}{6},\ \dfrac{11}{6}\pi$

Therefore,

$\dfrac{\pi}{6} < \theta < \dfrac{11}{6}\pi$

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(2) $\cos 2\theta - 3\sin\theta - 2 > 0$

[Sol] $(1-2\sin^2\theta) - 3\sin\theta - 2 > 0$

$2\sin^2\theta + 3\sin\theta + 1 < 0$

$(\sin\theta + 1)(2\sin\theta + 1) < 0$

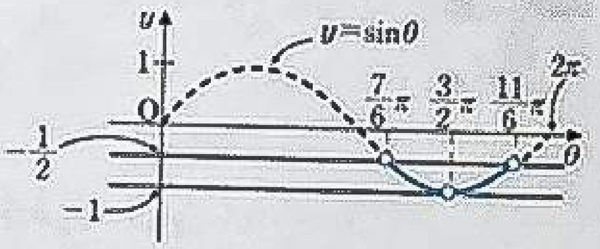
$\therefore\ -1 < \sin\theta < -\frac{1}{2}$

When $\sin\theta = -1,\ \theta = \frac{3}{2}\pi$

When $\sin\theta = -\frac{1}{2},\ \theta = \frac{7}{6}\pi,\ \frac{11}{6}\pi$

Therefore,

$\frac{7}{6}\pi < \theta < \frac{3}{2}\pi,\ \frac{3}{2}\pi < \theta < \frac{11}{6}\pi$



(3) $\sin\theta \geqq \cos 2\theta$

[Sol] $\sin\theta \geqq 1 - 2\sin^2\theta$

$2\sin^2\theta + \sin\theta - 1 \geqq 0$

$(\sin\theta + 1)(2\sin\theta - 1) \geqq 0$

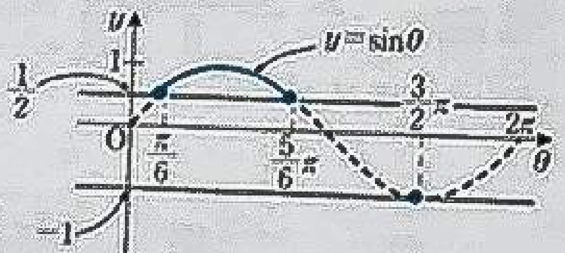
$\therefore\ \sin\theta \leqq -1,\ \frac{1}{2} \leqq \sin\theta$

When $\sin\theta = -1,\ \theta = \frac{3}{2}\pi$

When $\sin\theta = \frac{1}{2},\ \theta = \frac{\pi}{6},\ \frac{5}{6}\pi$

Therefore,

$\frac{\pi}{6} \leqq \theta \leqq \frac{5}{6}\pi,\ \theta = \frac{3}{2}\pi$



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Addition Formulas 2

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Given $0 \leqslant \theta < 2\pi$, find the maximum and minimum values of each given functi[on] and state the corresponding values of θ.

Ex. $y = \cos 2\theta - 2\cos\theta$

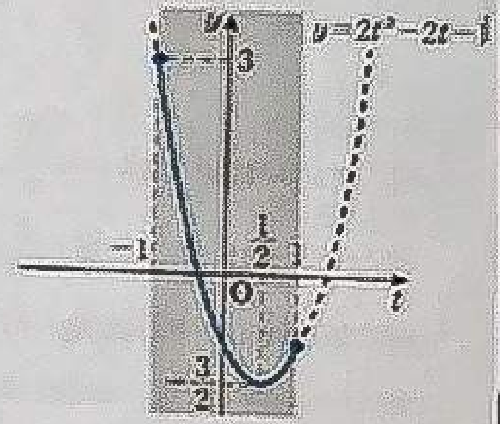
[Sol] $y = (2\cos^2\theta - 1) - 2\cos\theta$

$= 2\cos^2\theta - 2\cos\theta - 1$

Let $\cos\theta = t$, $-1 \leqslant t \leqslant 1$

$y = 2t^2 - 2t - 1$

$= 2\left(t - \frac{1}{2}\right)^2 - \frac{3}{2}$



Therefore,
the maximum value is 3,
at $t = \cos\theta = -1$, i.e. $\theta = \pi$ and
the minimum value is $-\frac{3}{2}$,
at $t = \cos\theta = \frac{1}{2}$, i.e. $\theta = \frac{\pi}{3}, \frac{5}{3}\pi$.

(1) $y = 2\sin\theta - \cos 2\theta$

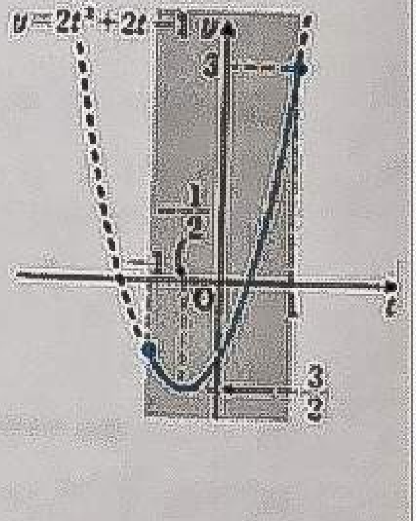
[Sol] $y = 2\sin\theta - (1 - 2\sin^2\theta)$

$= 2\sin^2\theta + 2\sin\theta - 1$

Let $\sin\theta = t$, $-1 \leqslant t \leqslant 1$

$y = 2t^2 + 2t - 1$

$= 2\left(t + \frac{1}{2}\right)^2 - \frac{3}{2}$



Therefore,
the maximum value is 3,
at $t = \sin\theta = 1$, i.e. $\theta = \frac{\pi}{2}$ and
the minimum value is $-\frac{3}{2}$,
at $t = \sin\theta = -\frac{1}{2}$, i.e. $\theta = \frac{7}{6}\pi, \frac{11}{6}\pi$.

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(2) $y=-\frac{1}{2}\cos 2\theta+\cos\theta+\frac{1}{2}$

[Sol] $y=-\frac{1}{2}(2\cos^2\theta-1)+\cos\theta+\frac{1}{2}$

$=-\cos^2\theta+\cos\theta+1$

Let $\cos\theta=t$. $-1\leqq t\leqq 1$

$y=-t^2+t+1$

$=-\left(t-\frac{1}{2}\right)^2+\frac{5}{4}$

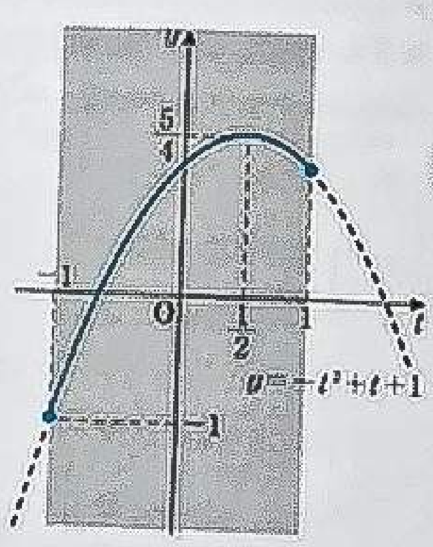
Therefore,

the maximum value is $\frac{5}{4}$,

at $t=\cos\theta=\frac{1}{2}$, i.e. $\theta=\frac{\pi}{3}$, $\frac{5}{3}\pi$ and

the minimum value is -1,

at $t=\cos\theta=-1$, i.e. $\theta=\pi$.



(3) $y=-\cos\theta+2\sin\frac{\theta}{2}$

[Sol] $y=-\cos\left(2\cdot\frac{\theta}{2}\right)+2\sin\frac{\theta}{2}$

$=-\left(1-2\sin^2\frac{\theta}{2}\right)+2\sin\frac{\theta}{2}$ ← $\cos\left(2\cdot\frac{\theta}{2}\right)=1-2\sin^2\frac{\theta}{2}$

$=2\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}-1$

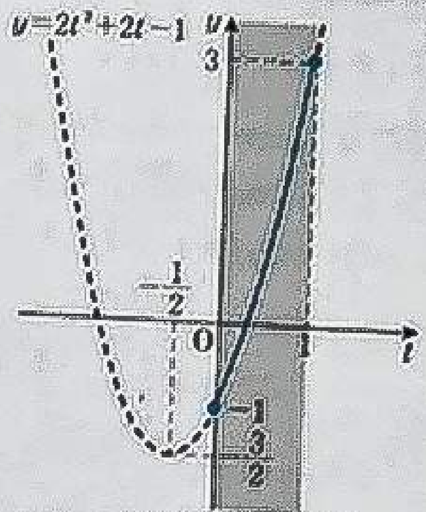
Let $\sin\frac{\theta}{2}=t$.

Since $0\leqq\theta<2\pi$, $0\leqq\frac{\theta}{2}<\pi$

$\therefore\ 0\leqq t\leqq 1$

$y=2t^2+2t-1$

$=2\left(t+\frac{1}{2}\right)^2-\frac{3}{2}$



Therefore,

the maximum value is 3,

at $t=\sin\frac{\theta}{2}=1$, i.e. $\theta=\pi$ and ← $\frac{\theta}{2}=\frac{\pi}{2}$

the minimum value is -1,

at $t=\sin\frac{\theta}{2}=0$, i.e. $\theta=0$. ← $\frac{\theta}{2}=0$

Addition Formulas 2

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1. Find the value of $\cos^8\theta-\sin^8\theta$ when $\cos 2\theta=\dfrac{1}{\sqrt{3}}$.

[Sol] $\cos^2\theta=\dfrac{1+\cos 2\theta}{2}=\dfrac{1+\dfrac{1}{\sqrt{3}}}{2}=\dfrac{3+\sqrt{3}}{6}$ ···① ⟵ Since $\cos 2\alpha=2\cos^2\alpha-1$, $\cos^2\alpha=\dfrac{1+\cos 2\alpha}{2}$

$\sin^2\theta=\dfrac{1-\cos 2\theta}{2}=\dfrac{1-\dfrac{1}{\sqrt{3}}}{2}=\dfrac{3-\sqrt{3}}{6}$ ···② ⟵ Since $\cos 2\alpha=1-2\sin^2\alpha$, $\sin^2\alpha=\dfrac{1-\cos 2\alpha}{2}$

From ①,

$$\cos^4\theta=\left(\frac{3+\sqrt{3}}{6}\right)^2=\frac{12+6\sqrt{3}}{36}=\frac{2+\sqrt{3}}{6}$$

From ②,

$$\sin^4\theta=\left(\frac{3-\sqrt{3}}{6}\right)^2=\frac{12-6\sqrt{3}}{36}=\frac{2-\sqrt{3}}{6}$$

Therefore,

$\cos^8\theta-\sin^8\theta=(\cos^4\theta+\sin^4\theta)(\cos^4\theta-\sin^4\theta)$ ⟵ $x^2-y^2=(x+y)(x-y)$

$$=\left(\frac{2+\sqrt{3}}{6}+\frac{2-\sqrt{3}}{6}\right)\left(\frac{2+\sqrt{3}}{6}-\frac{2-\sqrt{3}}{6}\right)$$

$$=\frac{2}{3}\cdot\frac{\sqrt{3}}{3}$$

$$=\frac{2\sqrt{3}}{9}$$

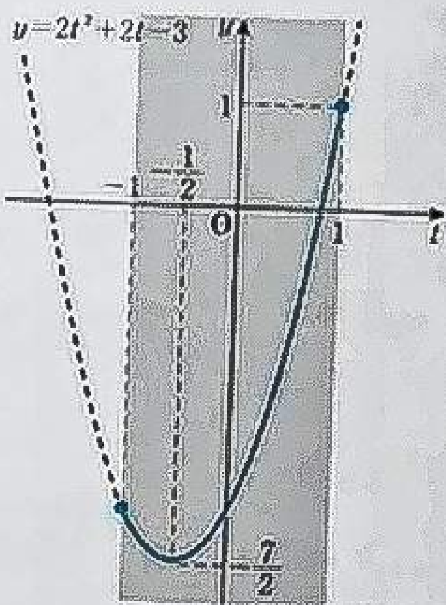
M169b

2. Given $0 \leqslant \theta \leqslant \frac{3}{4}\pi$, find the maximum and minimum values of $y = \cos 4\theta - 4\sin^2\theta$ and state the corresponding values of θ.

[Sol] $\cos 4\theta = 2\cos^2 2\theta - 1$

Also, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$\therefore\ y = (2\cos^2 2\theta - 1) - 4 \cdot \frac{1-\cos 2\theta}{2}$$
$$= 2\cos^2 2\theta + 2\cos 2\theta - 3$$



Let $\cos 2\theta = t$.

Since $0 \leqslant \theta \leqslant \frac{3}{4}\pi$, $0 \leqslant 2\theta \leqslant \frac{3}{2}\pi$

$\therefore\ -1 \leqslant t \leqslant 1$

$$y = 2t^2 + 2t - 3$$
$$= 2\left(t + \frac{1}{2}\right)^2 - \frac{7}{2}$$

Therefore,

the maximum value is 1,

at $t = \cos 2\theta = 1$, i.e. $\theta = 0$ and ⟵ $\boxed{2\theta = 0}$

the minimum value is $-\frac{7}{2}$,

at $t = \cos 2\theta = -\frac{1}{2}$, i.e. $\theta = \frac{\pi}{3}, \frac{2}{3}\pi$. ⟵ $\boxed{2\theta = \frac{2}{3}\pi, \frac{4}{3}\pi}$

Alternative Solution

$$\cos 4\theta = 1 - 2\sin^2 2\theta = 1 - 8\sin^2\theta\cos^2\theta$$
$$= 1 - 8\sin^2\theta(1 - \sin^2\theta) = 8\sin^4\theta - 8\sin^2\theta + 1$$
$$\therefore\ y = (8\sin^4\theta - 8\sin^2\theta + 1) - 4\sin^2\theta$$
$$= 8\sin^4\theta - 12\sin^2\theta + 1$$

Let $\sin^2\theta = t$.

Since $0 \leqslant \theta \leqslant \frac{3}{4}\pi$, $0 \leqslant t \leqslant 1$

$$y = 8t^2 - 12t + 1 = 8\left(t - \frac{3}{4}\right)^2 - \frac{7}{2}$$

Therefore, the maximum value is 1, at $t = \sin^2\theta = 0$, i.e. $\theta = 0$ and the minimum value is $-\frac{7}{2}$, at $t = \sin^2\theta = \frac{3}{4}$, i.e. $\theta = \frac{\pi}{3}, \frac{2}{3}\pi$.

M I 70a

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M I 70

Addition Formulas 2

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1. Let $\frac{\pi}{2}<\alpha<\pi$. Find the values of $\cos 2\alpha$ and $\sin 2\alpha$ when $\sin\alpha=\frac{2}{3}$.

➡ M I 6

[Sol] $\cos 2\alpha=1-2\cdot\left(\frac{2}{3}\right)^2=\frac{1}{9}$

Also, since $\frac{\pi}{2}<\alpha<\pi$, $\cos\alpha<0$

$$\cos\alpha=-\sqrt{1-\left(\frac{2}{3}\right)^2}=-\frac{\sqrt{5}}{3}$$

$$\therefore\ \sin 2\alpha=2\cdot\frac{2}{3}\cdot\left(-\frac{\sqrt{5}}{3}\right)=-\frac{4\sqrt{5}}{9}$$

2. Let $\pi<\alpha<\frac{3}{2}\pi$. Find the value of $\cos\frac{\alpha}{2}$ when $\sin\alpha=-\frac{\sqrt{5}}{3}$.

➡ M I 65

[Sol] Since $\pi<\alpha<\frac{3}{2}\pi$, $\cos\alpha<0$

$$\cos\alpha=-\sqrt{1-\left(-\frac{\sqrt{5}}{3}\right)^2}=-\frac{2}{3}$$

$$\therefore\ \cos^2\frac{\alpha}{2}=\frac{1+\left(-\frac{2}{3}\right)}{2}=\frac{1}{6}$$

Since $\pi<\alpha<\frac{3}{2}\pi$, $\frac{\pi}{2}<\frac{\alpha}{2}<\frac{3}{4}\pi$; therefore, $\cos\frac{\alpha}{2}<0$

$$\therefore\ \cos\frac{\alpha}{2}=-\frac{\sqrt{6}}{6}$$

3. Given $0\leqq\theta<2\pi$, solve the equation $\cos 2\theta+\sin\theta-1=0$.

➡ M I 66

[Sol] $(1-2\sin^2\theta)+\sin\theta-1=0$

$2\sin^2\theta-\sin\theta=0$

$\sin\theta(2\sin\theta-1)=0$

$\sin\theta=0,\ \frac{1}{2}$

$\therefore\ \theta=0,\ \frac{\pi}{6},\ \frac{5}{6}\pi,\ \pi$

4. Given $0 \leqslant \theta < 2\pi$, solve the inequality $\cos 2\theta - 3\cos\theta - 1 > 0$. ➡ M167

[Sol] $(2\cos^2\theta - 1) - 3\cos\theta - 1 > 0$

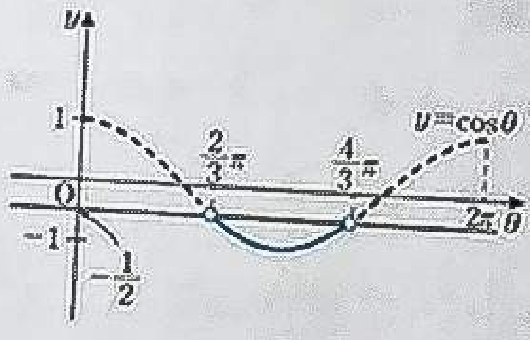
$2\cos^2\theta - 3\cos\theta - 2 > 0$

$(2\cos\theta + 1)(\cos\theta - 2) > 0$

Since $-1 \leqslant \cos\theta \leqslant 1$, always $\cos\theta - 2 < 0$.

$\therefore\ 2\cos\theta + 1 < 0$

$\cos\theta < -\frac{1}{2}$



When $\cos\theta = -\frac{1}{2}$, $\theta = \frac{2}{3}\pi,\ \frac{4}{3}\pi$

Therefore,

$\frac{2}{3}\pi < \theta < \frac{4}{3}\pi$

5. Given $0 \leqslant \theta < 2\pi$, find the maximum and minimum values of function $y = 2\sin\theta + \cos 2\theta + 1$ and state the corresponding values of θ. ➡ M168

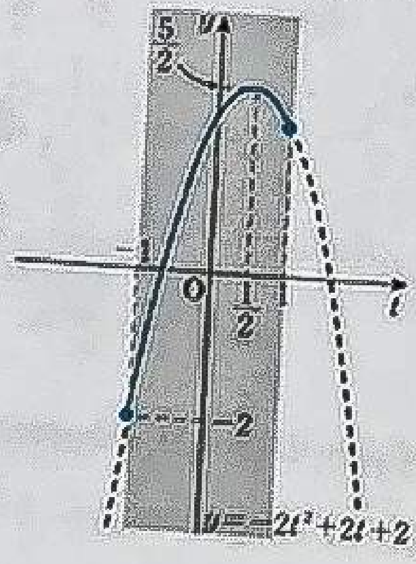
[Sol] $y = 2\sin\theta + (1 - 2\sin^2\theta) + 1$

$= -2\sin^2\theta + 2\sin\theta + 2$

Let $\sin\theta = t$. $-1 \leqslant t \leqslant 1$

$y = -2t^2 + 2t + 2$

$= -2\left(t - \frac{1}{2}\right)^2 + \frac{5}{2}$



Therefore,

the maximum value is $\frac{5}{2}$,

at $t = \sin\theta = \frac{1}{2}$, i.e. $\theta = \frac{\pi}{6},\ \frac{5}{6}\pi$ and

the minimum value is -2,

at $t = \sin\theta = -1$, i.e. $\theta = \frac{3}{2}\pi$.

M I 71 a

Addition Formulas 3

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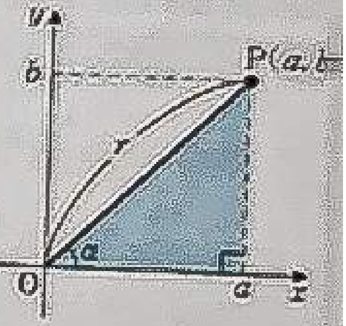
Using the Addition Formulas, convert $a\sin\theta+b\cos\theta$ into the form $r\sin(\theta+\alpha)$. Fill in the following blanks.

[Sol] Placing point $P(a, b)$ on a coordinate plane, let α be the angle formed by line segment OP and the positive x-axis. Also, let $OP=r$.



$\sin\alpha=\frac{b}{r}$, $\cos\alpha=\frac{a}{r}$

So, $a=\boxed{r\cos\alpha}$ ⟵ $\cos\alpha=\frac{a}{r}$

$b=\boxed{r\sin\alpha}$ ⟵ $\sin\alpha=\frac{b}{r}$

Therefore,

$$a\sin\theta+b\cos\theta=\boxed{r\cos\alpha}\sin\theta+\boxed{r\sin\alpha}\cos\theta$$
$$=r\sin(\theta+\alpha)$$

⟵ $\sin\theta\cos\alpha+\cos\theta\sin\alpha=\sin(\theta+\alpha)$

Also, $r=\boxed{\sqrt{a^2+b^2}}$

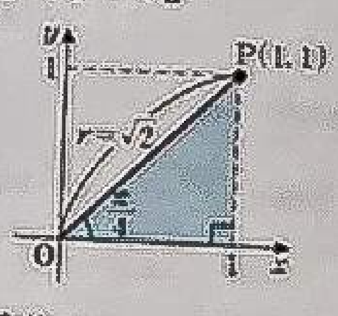
$$\therefore\ a\sin\theta+b\cos\theta=\boxed{\sqrt{a^2+b^2}}\sin(\theta+\alpha)$$

Answers: $r\cos\alpha$, $r\sin\alpha$, $r\cos\alpha$, $r\sin\alpha$, $\sqrt{a^2+b^2}$, $\sqrt{a^2+b^2}$

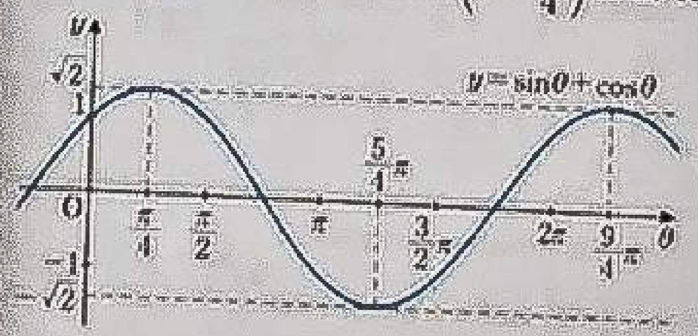
Consider converting $\sin\theta+\cos\theta$ into the form $r\sin(\theta+\alpha)$.

Placing point $P(1, 1)$ on a coordinate plane, $OP=\sqrt{1^2+1^2}=\sqrt{2}$

$$\sin\theta+\cos\theta=\sqrt{2}\left(\frac{\sqrt{2}}{2}\sin\theta+\frac{\sqrt{2}}{2}\cos\theta\right)$$
$$=\sqrt{2}\left(\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}\right)$$
$$=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$$



The graph of $y=\sin\theta+\cos\theta=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$ is as follows.



⟵ M I 39

M171b

Conversion of $a\sin\theta+b\cos\theta$

$$a\sin\theta+b\cos\theta=\sqrt{a^2+b^2}\sin(\theta+\alpha)$$

where $\cos\alpha=\dfrac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha=\dfrac{b}{\sqrt{a^2+b^2}}$

Convert the following expressions into the form $r\sin(\theta+\alpha)$. $(-\pi<\alpha<\pi)$

Ex $\sin\theta+\sqrt{3}\cos\theta$

[Sol] Since $\sqrt{1^2+(\sqrt{3})^2}=2$,

$$\sin\theta+\sqrt{3}\cos\theta=2\left(\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta\right)$$
$$=2\left(\sin\theta\cos\frac{\pi}{3}+\cos\theta\sin\frac{\pi}{3}\right)$$
$$=2\sin\left(\theta+\frac{\pi}{3}\right)$$

$r=\sqrt{1^2+(\sqrt{3})^2}$

$\frac{1}{2}=\cos\frac{\pi}{3}$

$\frac{\sqrt{3}}{2}=\sin\frac{\pi}{3}$

$\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$=\sin(\alpha+\beta)$

(1) $\sqrt{3}\sin\theta+\cos\theta$

[Sol] Since $\sqrt{(\sqrt{3})^2+1^2}=2$,

$$\sqrt{3}\sin\theta+\cos\theta=2\left(\frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta\right)$$
$$=2\left(\sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}\right)$$
$$=2\sin\left(\theta+\frac{\pi}{6}\right)$$

(2) $\sin\theta-\cos\theta$

[Sol] Since $\sqrt{1^2+(-1)^2}=\sqrt{2}$,

$$\sin\theta-\cos\theta=\sqrt{2}\left(\frac{\sqrt{2}}{2}\sin\theta-\frac{\sqrt{2}}{2}\cos\theta\right)$$
$$=\sqrt{2}\left[\sin\theta\cos\left(-\frac{\pi}{4}\right)+\cos\theta\sin\left(-\frac{\pi}{4}\right)\right]$$
$$=\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)$$

The results can be verified using the Addition Formulas.

In the case of **Ex**, $2\sin\left(\theta+\frac{\pi}{3}\right)=2\left(\sin\theta\cos\frac{\pi}{3}+\cos\theta\sin\frac{\pi}{3}\right)=\sin\theta+\sqrt{3}\cos\theta$

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Addition Formulas 3

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Given $0 \leqq \theta < 2\pi$, solve the following equations.

Ex. $\sin\theta - \sqrt{3}\cos\theta = 1$

[Sol] $\sin\theta - \sqrt{3}\cos\theta = 2\left(\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta\right) \leftarrow \boxed{\sqrt{1^2+(-\sqrt{3})^2}=2}$

$= 2\left[\sin\theta\cos\left(-\frac{\pi}{3}\right) + \cos\theta\sin\left(-\frac{\pi}{3}\right)\right]$

$= 2\sin\left(\theta - \frac{\pi}{3}\right)$

$\boxed{\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta)}$

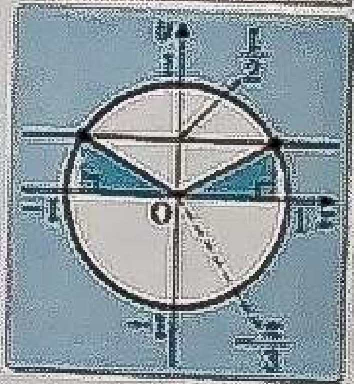
$\therefore\ 2\sin\left(\theta - \frac{\pi}{3}\right) = 1$

So, $\sin\left(\theta - \frac{\pi}{3}\right) = \frac{1}{2}$

Since $0 \leqq \theta < 2\pi$, $-\frac{\pi}{3} \leqq \theta - \frac{\pi}{3} < \frac{5}{3}\pi$

$\theta - \frac{\pi}{3} = \frac{\pi}{6},\ \frac{5}{6}\pi$

$\therefore\ \theta = \frac{\pi}{2},\ \frac{7}{6}\pi$



(1) $\sqrt{3}\sin\theta + \cos\theta = 0$

[Sol] $\sqrt{3}\sin\theta + \cos\theta = 2\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)$

$= 2\left(\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right)$

$= 2\sin\left(\theta + \frac{\pi}{6}\right)$

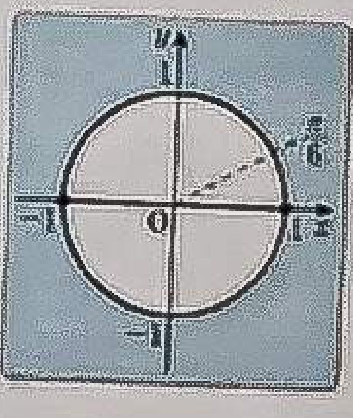
$\therefore\ 2\sin\left(\theta + \frac{\pi}{6}\right) = 0$

So, $\sin\left(\theta + \frac{\pi}{6}\right) = 0$

Since $0 \leqq \theta < 2\pi$, $\frac{\pi}{6} \leqq \theta + \frac{\pi}{6} < \frac{13}{6}\pi$

$\theta + \frac{\pi}{6} = \pi,\ 2\pi$

$\therefore\ \theta = \frac{5}{6}\pi,\ \frac{11}{6}\pi$



M172b

(2) $\sin\theta-\cos\theta=-\dfrac{\sqrt{2}}{2}$

[Sol] $\sin\theta-\cos\theta=\sqrt{2}\left(\dfrac{\sqrt{2}}{2}\sin\theta-\dfrac{\sqrt{2}}{2}\cos\theta\right)$

$$=\sqrt{2}\left[\sin\theta\cos\left(-\frac{\pi}{4}\right)+\cos\theta\sin\left(-\frac{\pi}{4}\right)\right]$$

$$=\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)$$

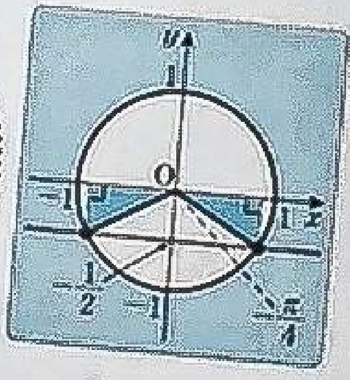
$\therefore\ \sqrt{2}\sin\left(\theta-\dfrac{\pi}{4}\right)=-\dfrac{\sqrt{2}}{2}$

So, $\sin\left(\theta-\dfrac{\pi}{4}\right)=-\dfrac{1}{2}$

Since $0\leqq\theta<2\pi$, $-\dfrac{\pi}{4}\leqq\theta-\dfrac{\pi}{4}<\dfrac{7}{4}\pi$

$\theta-\dfrac{\pi}{4}=-\dfrac{\pi}{6},\ \dfrac{7}{6}\pi$

$\therefore\ \theta=\dfrac{\pi}{12},\ \dfrac{17}{12}\pi$



(3) $\sqrt{2}\sin\theta-\sqrt{6}\cos\theta=2$

[Sol] $\sqrt{2}\sin\theta-\sqrt{6}\cos\theta=2\sqrt{2}\left(\dfrac{1}{2}\sin\theta-\dfrac{\sqrt{3}}{2}\cos\theta\right)$

$$=2\sqrt{2}\left[\sin\theta\cos\left(-\frac{\pi}{3}\right)+\cos\theta\sin\left(-\frac{\pi}{3}\right)\right]$$

$$=2\sqrt{2}\sin\left(\theta-\frac{\pi}{3}\right)$$

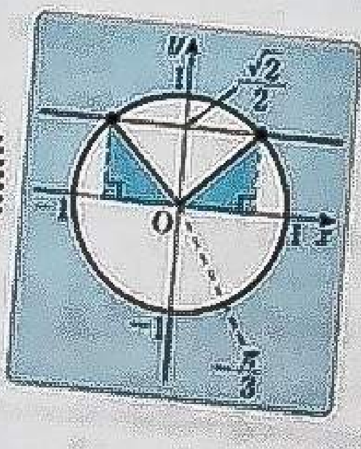
$\therefore\ 2\sqrt{2}\sin\left(\theta-\dfrac{\pi}{3}\right)=2$

So, $\sin\left(\theta-\dfrac{\pi}{3}\right)=\dfrac{\sqrt{2}}{2}$

Since $0\leqq\theta<2\pi$, $-\dfrac{\pi}{3}\leqq\theta-\dfrac{\pi}{3}<\dfrac{5}{3}\pi$

$\theta-\dfrac{\pi}{3}=\dfrac{\pi}{4},\ \dfrac{3}{4}\pi$

$\therefore\ \theta=\dfrac{7}{12}\pi,\ \dfrac{13}{12}\pi$



Addition Formulas 3

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Given $0 \leqq \theta < 2\pi$, solve the following inequalities.

Ex. $\sin\theta + \sqrt{3}\cos\theta < 1$

[Sol] $\sin\theta + \sqrt{3}\cos\theta = 2\left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right)$ ⟵ $\boxed{\sqrt{1+(\sqrt{3})^2}=2}$

$= 2\left(\sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3}\right)$

$= 2\sin\left(\theta + \frac{\pi}{3}\right)$ ⟵ $\boxed{\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta)}$

Therefore, $2\sin\left(\theta+\frac{\pi}{3}\right) < 1$, i.e. $\sin\left(\theta+\frac{\pi}{3}\right) < \frac{1}{2}$

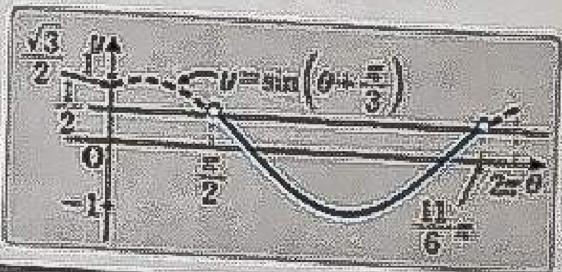
Since $0 \leqq \theta < 2\pi$, $\frac{\pi}{3} \leqq \theta + \frac{\pi}{3} < \frac{7}{3}\pi$

When $\sin\left(\theta+\frac{\pi}{3}\right) = \frac{1}{2}$,

$\theta + \frac{\pi}{3} = \frac{5}{6}\pi,\ \frac{13}{6}\pi$

$\therefore\ \theta = \frac{\pi}{2},\ \frac{11}{6}\pi$

$\therefore\ \frac{\pi}{2} < \theta < \frac{11}{6}\pi$ ⟵



(1) $\sin\theta - \cos\theta > 1$

[Sol] $\sin\theta - \cos\theta = \sqrt{2}\left(\frac{\sqrt{2}}{2}\sin\theta - \frac{\sqrt{2}}{2}\cos\theta\right)$

$= \sqrt{2}\left[\sin\theta\cos\left(-\frac{\pi}{4}\right) + \cos\theta\sin\left(-\frac{\pi}{4}\right)\right]$

$= \sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right)$

Therefore, $\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right) > 1$, i.e. $\sin\left(\theta-\frac{\pi}{4}\right) > \frac{\sqrt{2}}{2}$

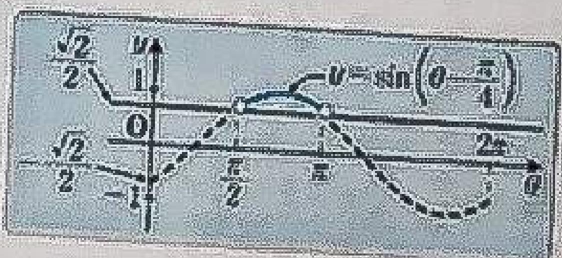
Since $0 \leqq \theta < 2\pi$, $-\frac{\pi}{4} \leqq \theta - \frac{\pi}{4} < \frac{7}{4}\pi$

When $\sin\left(\theta-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$,

$\theta - \frac{\pi}{4} = \frac{\pi}{4},\ \frac{3}{4}\pi$

$\therefore\ \theta = \frac{\pi}{2},\ \pi$

$\therefore\ \frac{\pi}{2} < \theta < \pi$ ⟵



M173b

(2) $\sqrt{3}\sin\theta-\cos\theta\leqq\sqrt{2}$

[Sol] $\sqrt{3}\sin\theta-\cos\theta=2\left(\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta\right)$

$=2\left[\sin\theta\cos\left(-\frac{\pi}{6}\right)+\cos\theta\sin\left(-\frac{\pi}{6}\right)\right]$

$=2\sin\left(\theta-\frac{\pi}{6}\right)$

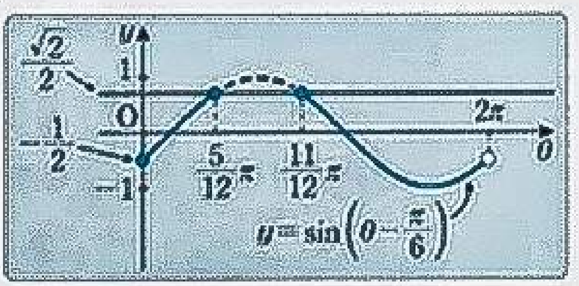
Therefore, $2\sin\left(\theta-\frac{\pi}{6}\right)\leqq\sqrt{2}$, i.e. $\sin\left(\theta-\frac{\pi}{6}\right)\leqq\frac{\sqrt{2}}{2}$

Since $0\leqq\theta<2\pi$, $-\frac{\pi}{6}\leqq\theta-\frac{\pi}{6}<\frac{11}{6}\pi$

When $\sin\left(\theta-\frac{\pi}{6}\right)=\frac{\sqrt{2}}{2}$,

$\theta-\frac{\pi}{6}=\frac{\pi}{4}, \frac{3}{4}\pi$

$\therefore\ \theta=\frac{5}{12}\pi, \frac{11}{12}\pi$



$\therefore\ 0\leqq\theta\leqq\frac{5}{12}\pi,\ \frac{11}{12}\pi\leqq\theta<2\pi$

(3) $\sqrt{3}\sin2\theta+\cos2\theta>-1$

[Sol] $\sqrt{3}\sin2\theta+\cos2\theta=2\left(\frac{\sqrt{3}}{2}\sin2\theta+\frac{1}{2}\cos2\theta\right)$

$=2\left(\sin2\theta\cos\frac{\pi}{6}+\cos2\theta\sin\frac{\pi}{6}\right)$

$=2\sin\left(2\theta+\frac{\pi}{6}\right)$

Therefore, $2\sin\left(2\theta+\frac{\pi}{6}\right)>-1$, i.e. $\sin\left(2\theta+\frac{\pi}{6}\right)>-\frac{1}{2}$

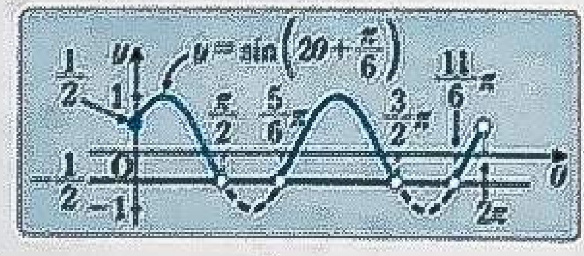
Since $0\leqq\theta<2\pi$, $\frac{\pi}{6}\leqq2\theta+\frac{\pi}{6}<\frac{25}{6}\pi$

When $\sin\left(2\theta+\frac{\pi}{6}\right)=-\frac{1}{2}$,

$2\theta+\frac{\pi}{6}=\frac{7}{6}\pi, \frac{11}{6}\pi, \frac{19}{6}\pi, \frac{23}{6}\pi$

$\therefore\ 2\theta=\pi, \frac{5}{3}\pi, 3\pi, \frac{11}{3}\pi$

$\therefore\ \theta=\frac{\pi}{2}, \frac{5}{6}\pi, \frac{3}{2}\pi, \frac{11}{6}\pi$



$\therefore\ 0\leqq\theta<\frac{\pi}{2},\ \frac{5}{6}\pi<\theta<\frac{3}{2}\pi,\ \frac{11}{6}\pi<\theta<2\pi$

Addition Formulas 3

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Find the maximum and minimum values of each given function and state the corresponding values of θ.

Ex. $y=\sin\theta-\cos\theta \quad (0\leqq\theta<2\pi)$

[Sol] $y=\sqrt{2}\left(\frac{\sqrt{2}}{2}\sin\theta-\frac{\sqrt{2}}{2}\cos\theta\right)$

$=\sqrt{2}\left[\sin\theta\cos\left(-\frac{\pi}{4}\right)+\cos\theta\sin\left(-\frac{\pi}{4}\right)\right]$

$=\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)$

Since $0\leqq\theta<2\pi$, $-\frac{\pi}{4}\leqq\theta-\frac{\pi}{4}<\frac{7}{4}\pi$; therefore, $-1\leqq\sin\left(\theta-\frac{\pi}{4}\right)\leqq1$

$\therefore\ -\sqrt{2}\leqq y\leqq\sqrt{2}$

Also, when $\sin\left(\theta-\frac{\pi}{4}\right)=1$, $\theta-\frac{\pi}{4}=\frac{\pi}{2}$, i.e. $\theta=\frac{3}{4}\pi$

when $\sin\left(\theta-\frac{\pi}{4}\right)=-1$, $\theta-\frac{\pi}{4}=\frac{3}{2}\pi$, i.e. $\theta=\frac{7}{4}\pi$

Therefore, the maximum value is $\sqrt{2}$, at $\theta=\frac{3}{4}\pi$ and

the minimum value is $-\sqrt{2}$, at $\theta=\frac{7}{4}\pi$.

(1) $y=\sin\theta+\sqrt{3}\cos\theta \quad (0\leqq\theta<2\pi)$

[Sol] $y=2\left(\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta\right)$

$=2\left(\sin\theta\cos\frac{\pi}{3}+\cos\theta\sin\frac{\pi}{3}\right)$

$=2\sin\left(\theta+\frac{\pi}{3}\right)$

Since $0\leqq\theta<2\pi$, $\frac{\pi}{3}\leqq\theta+\frac{\pi}{3}<\frac{7}{3}\pi$; therefore, $-1\leqq\sin\left(\theta+\frac{\pi}{3}\right)\leqq1$

$\therefore\ -2\leqq y\leqq2$

Also, when $\sin\left(\theta+\frac{\pi}{3}\right)=1$, $\theta+\frac{\pi}{3}=\frac{\pi}{2}$, i.e. $\theta=\frac{\pi}{6}$

when $\sin\left(\theta+\frac{\pi}{3}\right)=-1$, $\theta+\frac{\pi}{3}=\frac{3}{2}\pi$, i.e. $\theta=\frac{7}{6}\pi$

Therefore, the maximum value is 2, at $\theta=\frac{\pi}{6}$ and

the minimum value is -2, at $\theta=\frac{7}{6}\pi$.

M174b

(2) $y=2\sqrt{3}\sin\theta\cos\theta+\cos 2\theta \quad (0\leqq\theta<\pi)$

Since $\sin 2\alpha=2\sin\alpha\cos\alpha$, $\sin\alpha\cos\alpha=\frac{1}{2}\sin 2\alpha$

[Sol] $y=\sqrt{3}\sin 2\theta+\cos 2\theta$

$$=2\left(\frac{\sqrt{3}}{2}\sin 2\theta+\frac{1}{2}\cos 2\theta\right)$$

$$=2\left(\sin 2\theta\cos\frac{\pi}{6}+\cos 2\theta\sin\frac{\pi}{6}\right)$$

$$=2\sin\left(2\theta+\frac{\pi}{6}\right)$$

Since $0\leqq\theta<\pi$, $\frac{\pi}{6}\leqq 2\theta+\frac{\pi}{6}<\frac{13}{6}\pi$; therefore, $-1\leqq\sin\left(2\theta+\frac{\pi}{6}\right)\leqq 1$

$\therefore\ -2\leqq y\leqq 2$

Also, when $\sin\left(2\theta+\frac{\pi}{6}\right)=1$, $2\theta+\frac{\pi}{6}=\frac{\pi}{2}$, i.e. $\theta=\frac{\pi}{6}$

when $\sin\left(2\theta+\frac{\pi}{6}\right)=-1$, $2\theta+\frac{\pi}{6}=\frac{3}{2}\pi$, i.e. $\theta=\frac{2}{3}\pi$

Therefore, the maximum value is 2, at $\theta=\frac{\pi}{6}$ and

the minimum value is -2, at $\theta=\frac{2}{3}\pi$.

(3) $y=\sqrt{3}\sin\theta\cos\theta+\cos^2\theta \quad \left(0\leqq\theta\leqq\frac{\pi}{2}\right)$

Since $\sin 2\alpha=2\sin\alpha\cos\alpha$, $\sin\alpha\cos\alpha=\frac{1}{2}\sin 2\alpha$

Since $\cos 2\alpha=2\cos^2\alpha-1$, $\cos^2\alpha=\frac{1+\cos 2\alpha}{2}$

[Sol] $y=\frac{\sqrt{3}}{2}\sin 2\theta+\frac{1+\cos 2\theta}{2}$

$$=\frac{\sqrt{3}}{2}\sin 2\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}$$

$$=\sin 2\theta\cos\frac{\pi}{6}+\cos 2\theta\sin\frac{\pi}{6}+\frac{1}{2}$$

$$=\sin\left(2\theta+\frac{\pi}{6}\right)+\frac{1}{2}$$

When $2\theta+\frac{\pi}{6}=\frac{\pi}{2}$, the maximum value is 1.

When $2\theta+\frac{\pi}{6}=\frac{7}{6}\pi$, the minimum value is $-\frac{1}{2}$.

Since $0\leqq\theta\leqq\frac{\pi}{2}$, $\frac{\pi}{6}\leqq 2\theta+\frac{\pi}{6}\leqq\frac{7}{6}\pi$; therefore, $-\frac{1}{2}\leqq\sin\left(2\theta+\frac{\pi}{6}\right)\leqq 1$

$\therefore\ 0\leqq y\leqq\frac{3}{2}$

Also, when $\sin\left(2\theta+\frac{\pi}{6}\right)=1$, $2\theta+\frac{\pi}{6}=\frac{\pi}{2}$, i.e. $\theta=\frac{\pi}{6}$

when $\sin\left(2\theta+\frac{\pi}{6}\right)=-\frac{1}{2}$, $2\theta+\frac{\pi}{6}=\frac{7}{6}\pi$, i.e. $\theta=\frac{\pi}{2}$

Therefore, the maximum value is $\frac{3}{2}$, at $\theta=\frac{\pi}{6}$ and

the minimum value is 0, at $\theta=\frac{\pi}{2}$.

M I 75a

KUMON

Addition Formulas 3

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1. Prove the identities (1)~(3) using the Addition Formulas ①~④.

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ …①

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ …②

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ …③

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ …④

Ex.

$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

[Sol] From ①+②,

$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$

$\therefore\ \sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

(1) $\cos\alpha\sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

[Sol] From ①−②,

$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta$

$\therefore\ \cos\alpha\sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

(2) $\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$

[Sol] From ③+④,

$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta$

$\therefore\ \cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$

(3) $\sin\alpha\sin\beta=-\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$

[Sol] From ③−④,

$\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin\alpha\sin\beta$

$\therefore\ \sin\alpha\sin\beta=-\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$

M175b

Product-to-Sum Formulas

$$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$
$$\cos\alpha\sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$
$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$
$$\sin\alpha\sin\beta=-\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$$

2. Convert the following expressions into a sum or difference using the formulas above.

Ex.

$$\sin 4\theta\cos 2\theta=\frac{1}{2}[\sin(4\theta+2\theta)+\sin(4\theta-2\theta)]$$
$$=\frac{1}{2}\sin 6\theta+\frac{1}{2}\sin 2\theta$$

(1) $\sin 6\theta\cos\theta=\frac{1}{2}[\sin(6\theta+\theta)+\sin(6\theta-\theta)]$

$$=\frac{1}{2}\sin 7\theta+\frac{1}{2}\sin 5\theta$$

(2) $\cos 7\theta\sin\theta=\frac{1}{2}[\sin(7\theta+\theta)-\sin(7\theta-\theta)]$

$$=\frac{1}{2}\sin 8\theta-\frac{1}{2}\sin 6\theta$$

(3) $\cos 5\theta\cos 3\theta=\frac{1}{2}[\cos(5\theta+3\theta)+\cos(5\theta-3\theta)]$

$$=\frac{1}{2}\cos 8\theta+\frac{1}{2}\cos 2\theta$$

(4) $\sin 6\theta\sin 2\theta=-\frac{1}{2}[\cos(6\theta+2\theta)-\cos(6\theta-2\theta)]$

$$=-\frac{1}{2}\cos 8\theta+\frac{1}{2}\cos 4\theta$$

(5) $\sin 3\theta\cos 5\theta=\frac{1}{2}[\sin(3\theta+5\theta)+\sin(3\theta-5\theta)]$

$$=\frac{1}{2}\sin 8\theta+\frac{1}{2}\sin(-2\theta)$$
$$=\frac{1}{2}\sin 8\theta-\frac{1}{2}\sin 2\theta$$

$\boxed{\sin(-\theta)=-\sin\theta}$

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Addition Formulas 3

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Product-to-Sum Formulas

$$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$
$$\cos\alpha\sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$
$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$
$$\sin\alpha\sin\beta=-\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$$

Evaluate the following expressions using the formulas above.

Ex.

$$\sin 75^\circ\cos 15^\circ=\frac{1}{2}[\sin(75^\circ+15^\circ)+\sin(75^\circ-15^\circ)]$$
$$=\frac{1}{2}(\sin 90^\circ+\sin 60^\circ)$$
$$=\frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right)=\frac{2+\sqrt{3}}{4}$$

(1) $\sin 45^\circ\cos 15^\circ=\frac{1}{2}[\sin(45^\circ+15^\circ)+\sin(45^\circ-15^\circ)]$

$$=\frac{1}{2}(\sin 60^\circ+\sin 30^\circ)$$
$$=\frac{1}{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)=\frac{\sqrt{3}+1}{4}$$

(2) $\cos 75^\circ\sin 15^\circ=\frac{1}{2}[\sin(75^\circ+15^\circ)-\sin(75^\circ-15^\circ)]$

$$=\frac{1}{2}(\sin 90^\circ-\sin 60^\circ)$$
$$=\frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)=\frac{2-\sqrt{3}}{4}$$

M176b

(3) $\cos 45^\circ \cos 15^\circ = \frac{1}{2}[\cos(45^\circ + 15^\circ) + \cos(45^\circ - 15^\circ)]$

$$= \frac{1}{2}(\cos 60^\circ + \cos 30^\circ)$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = \frac{1+\sqrt{3}}{4}$$

(4) $\sin 45^\circ \sin 75^\circ = -\frac{1}{2}[\cos(45^\circ + 75^\circ) - \cos(45^\circ - 75^\circ)]$

$$= -\frac{1}{2}[\cos 120^\circ - \cos(-30^\circ)]$$

$$= -\frac{1}{2}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right) = \frac{1+\sqrt{3}}{4}$$

(5) $\cos 20^\circ \cos 40^\circ - \sin 80^\circ \sin 100^\circ$

$$= \frac{1}{2}[\cos(20^\circ + 40^\circ) + \cos(20^\circ - 40^\circ)] + \frac{1}{2}[\cos(80^\circ + 100^\circ) - \cos(80^\circ - 100^\circ)]$$

$$= \frac{1}{2}[\cos 60^\circ + \cos(-20^\circ)] + \frac{1}{2}[\cos 180^\circ - \cos(-20^\circ)]$$

$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot(-1) = -\frac{1}{4}$$

(6) $\sin 20^\circ \sin 40^\circ \sin 80^\circ$

$$= -\frac{1}{2}[\cos(20^\circ + 40^\circ) - \cos(20^\circ - 40^\circ)]\sin 80^\circ$$

$$= -\frac{1}{2}[\cos 60^\circ - \cos(-20^\circ)]\sin 80^\circ$$

$\boxed{\cos(-\theta) = \cos\theta}$

$$= -\frac{1}{2}\left(\frac{1}{2} - \cos 20^\circ\right)\sin 80^\circ$$

$$= -\frac{1}{4}\sin 80^\circ + \frac{1}{2}\cos 20^\circ \sin 80^\circ$$

$$= -\frac{1}{4}\sin 80^\circ + \frac{1}{2}\cdot\frac{1}{2}[\sin(20^\circ + 80^\circ) - \sin(20^\circ - 80^\circ)]$$

$$= -\frac{1}{4}\sin 80^\circ + \frac{1}{4}[\sin 100^\circ - \sin(-60^\circ)]$$

$$= -\frac{1}{4}\sin 80^\circ + \frac{1}{4}\sin 100^\circ - \frac{1}{4}\cdot\left(-\frac{\sqrt{3}}{2}\right)$$

$\boxed{\sin(180^\circ - \theta) = \sin\theta}$

$$= -\frac{1}{4}\sin 80^\circ + \frac{1}{4}\sin 80^\circ + \frac{\sqrt{3}}{8}$$

$$= \frac{\sqrt{3}}{8}$$

M177a

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Addition Formulas 3

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1. Given $\alpha+\beta=A$ and $\alpha-\beta=B$, prove the identities (1)~(3) using the ide ①~④.

$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$ …①

$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta$ …②

$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta$ …③

$\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin\alpha\sin\beta$ …④

Ex.

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

[Sol] Since $\alpha+\beta=A$ and $\alpha-\beta=B$, $\alpha=\frac{A+B}{2}$, $\beta=\frac{A-B}{2}$

Substituting them into ①,

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

(1) $\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

[Sol] Since $\alpha+\beta=A$ and $\alpha-\beta=B$, $\alpha=\frac{A+B}{2}$, $\beta=\frac{A-B}{2}$

Substituting them into ②,

$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

(2) $\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

[Sol] Since $\alpha+\beta=A$ and $\alpha-\beta=B$, $\alpha=\frac{A+B}{2}$, $\beta=\frac{A-B}{2}$

Substituting them into ③,

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

(3) $\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

[Sol] Since $\alpha+\beta=A$ and $\alpha-\beta=B$, $\alpha=\frac{A+B}{2}$, $\beta=\frac{A-B}{2}$

Substituting them into ④,

$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

M177b

Sum-to-Product Formulas

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

2. Convert the following expressions into a product using the formulas above.

Ex.

$$\sin 5\theta+\sin 3\theta=2\sin\frac{5\theta+3\theta}{2}\cos\frac{5\theta-3\theta}{2}$$
$$=2\sin 4\theta\cos\theta$$

(1) $\sin 7\theta+\sin\theta=2\sin\frac{7\theta+\theta}{2}\cos\frac{7\theta-\theta}{2}$

$=\mathbf{2\sin 4\theta\cos 3\theta}$

(2) $\sin 7\theta-\sin 3\theta=2\cos\frac{7\theta+3\theta}{2}\sin\frac{7\theta-3\theta}{2}$

$=\mathbf{2\cos 5\theta\sin 2\theta}$

(3) $\cos 5\theta-\cos 7\theta=-2\sin\frac{5\theta+7\theta}{2}\sin\frac{5\theta-7\theta}{2}$

$=-2\sin 6\theta\sin(-\theta)$

$=\mathbf{2\sin 6\theta\sin\theta}$ $\boxed{\sin(-\theta)=-\sin\theta}$

(4) $\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta$

$=2\cos\frac{\theta+3\theta}{2}\cos\frac{\theta-3\theta}{2}+2\cos\frac{5\theta+7\theta}{2}\cos\frac{5\theta-7\theta}{2}$

$=2\cos 2\theta\cos(-\theta)+2\cos 6\theta\cos(-\theta)$

$=2\cos\theta(\cos 2\theta+\cos 6\theta)$ $\boxed{\cos(-\theta)=\cos\theta}$

$=2\cos\theta\cdot 2\cos\frac{2\theta+6\theta}{2}\cos\frac{2\theta-6\theta}{2}$

$=4\cos\theta\cos 4\theta\cos(-2\theta)$

$=\mathbf{4\cos\theta\cos 4\theta\cos 2\theta}$ $\boxed{\cos(-2\theta)=\cos 2\theta}$

KUMON

Addition Formulas 3

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Sum-to-Product Formulas

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

1. Evaluate the following expressions using the formulas above.

(1) $\sin 75^\circ+\sin 15^\circ=2\sin\frac{75^\circ+15^\circ}{2}\cos\frac{75^\circ-15^\circ}{2}$

$=2\sin 45^\circ\cos 30^\circ$

$=2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{6}}{2}$

(2) $\sin 105^\circ-\sin 15^\circ=2\cos\frac{105^\circ+15^\circ}{2}\sin\frac{105^\circ-15^\circ}{2}$

$=2\cos 60^\circ\sin 45^\circ$

$=2\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}$

(3) $\cos 105^\circ-\cos 15^\circ=-2\sin\frac{105^\circ+15^\circ}{2}\sin\frac{105^\circ-15^\circ}{2}$

$=-2\sin 60^\circ\sin 45^\circ$

$=-2\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=-\frac{\sqrt{6}}{2}$

(4) $\cos 10^\circ+\cos 110^\circ+\cos 230^\circ=2\cos\frac{10^\circ+110^\circ}{2}\cos\frac{10^\circ-110^\circ}{2}+\cos 230^\circ$

$=2\cos 60^\circ\cos(-50^\circ)+\cos 230^\circ$

$\cos(-\theta)=\cos\theta$

$=\cos 50^\circ+\cos 230^\circ$

$=2\cos\frac{50^\circ+230^\circ}{2}\cos\frac{50^\circ-230^\circ}{2}$

$=2\cos 140^\circ\cos(-90^\circ)$

$=0$

M178b

2. Given $0 \leqslant \theta < \pi$, solve the following equations.

Ex. $\sin 2\theta + \sin 3\theta + \sin 4\theta = 0$

[Sol] $2\sin\frac{2\theta+4\theta}{2}\cos\frac{2\theta-4\theta}{2} + \sin 3\theta = 0$

$2\sin 3\theta\cos(-\theta) + \sin 3\theta = 0$

$2\sin 3\theta\cos\theta + \sin 3\theta = 0$

$\sin 3\theta(2\cos\theta + 1) = 0$

$\sin 3\theta = 0,\ \cos\theta = -\frac{1}{2}$

When $\sin 3\theta = 0$,

since $0 \leqslant \theta < \pi,\ 0 \leqslant 3\theta < 3\pi$

$\therefore\ 3\theta = 0,\ \pi,\ 2\pi \quad \therefore\ \theta = 0,\ \frac{\pi}{3},\ \frac{2}{3}\pi$

Also, when $\cos\theta = -\frac{1}{2}$,

since $0 \leqslant \theta < \pi,\ \theta = \frac{2}{3}\pi$

$\therefore\ \theta = 0,\ \frac{\pi}{3},\ \frac{2}{3}\pi$ ⟵ When $\theta = \frac{2}{3}\pi$, $\sin 3\theta = 0$ and also $\cos\theta = -\frac{1}{2}$.

(1) $\cos\theta + \cos 2\theta + \cos 3\theta = 0$

[Sol] $2\cos\frac{\theta+3\theta}{2}\cos\frac{\theta-3\theta}{2} + \cos 2\theta = 0$

$2\cos 2\theta\cos(-\theta) + \cos 2\theta = 0$

$2\cos 2\theta\cos\theta + \cos 2\theta = 0$

$\cos 2\theta(2\cos\theta + 1) = 0$

$\cos 2\theta = 0,\ \cos\theta = -\frac{1}{2}$

When $\cos 2\theta = 0$,

since $0 \leqslant \theta < \pi,\ 0 \leqslant 2\theta < 2\pi$

$\therefore\ 2\theta = \frac{\pi}{2},\ \frac{3}{2}\pi \quad \therefore\ \theta = \frac{\pi}{4},\ \frac{3}{4}\pi$

Also, when $\cos\theta = -\frac{1}{2}$,

since $0 \leqslant \theta < \pi,\ \theta = \frac{2}{3}\pi$

$\therefore\ \theta = \frac{\pi}{4},\ \frac{2}{3}\pi,\ \frac{3}{4}\pi$

Addition Formulas 3

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1. Given $0 \leqq \theta < 2\pi$, find the maximum and minimum values of the following functions.

Ex. $y=\sqrt{2}(\sin\theta+\cos\theta)-\sin\theta\cos\theta-1$

[Sol] Let $\sin\theta+\cos\theta=t$.

$\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta=t^2$

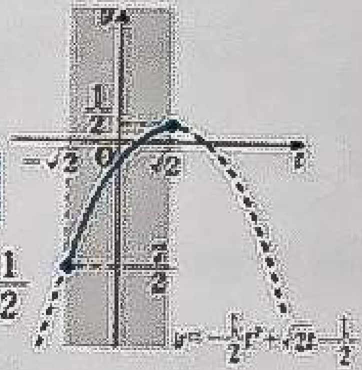
$1+2\sin\theta\cos\theta=t^2$

$\therefore\ \sin\theta\cos\theta=\dfrac{t^2-1}{2}$

Squaring both sides of $\sin\theta+\cos\theta=t$

$\therefore\ y=\sqrt{2}t-\dfrac{t^2-1}{2}-1=-\dfrac{1}{2}t^2+\sqrt{2}t-\dfrac{1}{2}$

$=-\dfrac{1}{2}(t-\sqrt{2})^2+\dfrac{1}{2}$



Also, $t=\sin\theta+\cos\theta=\sqrt{2}\left(\dfrac{\sqrt{2}}{2}\sin\theta+\dfrac{\sqrt{2}}{2}\cos\theta\right)$

$=\sqrt{2}\left(\sin\theta\cos\dfrac{\pi}{4}+\cos\theta\sin\dfrac{\pi}{4}\right)=\sqrt{2}\sin\left(\theta+\dfrac{\pi}{4}\right)$

$\therefore\ -\sqrt{2}\leqq t\leqq\sqrt{2}$

$\dfrac{\pi}{4}\leqq\theta+\dfrac{\pi}{4}<\dfrac{9}{4}\pi$

Therefore, the maximum value is $\dfrac{1}{2}$, at $t=\sqrt{2}$ and
the minimum value is $-\dfrac{7}{2}$, at $t=-\sqrt{2}$.

(1) $y=2\sin\theta\cos\theta-(\sin\theta+\cos\theta)+2$

[Sol] Let $\sin\theta+\cos\theta=t$.

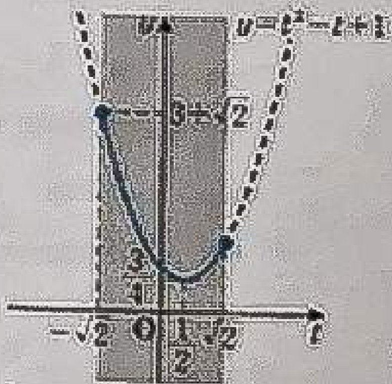
$\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta=t^2$

$1+2\sin\theta\cos\theta=t^2$

$\therefore\ \sin\theta\cos\theta=\dfrac{t^2-1}{2}$

$\therefore\ y=2\cdot\dfrac{t^2-1}{2}-t+2=t^2-t+1$

$=\left(t-\dfrac{1}{2}\right)^2+\dfrac{3}{4}$



Also, $t=\sin\theta+\cos\theta=\sqrt{2}\left(\dfrac{\sqrt{2}}{2}\sin\theta+\dfrac{\sqrt{2}}{2}\cos\theta\right)$

$=\sqrt{2}\left(\sin\theta\cos\dfrac{\pi}{4}+\cos\theta\sin\dfrac{\pi}{4}\right)=\sqrt{2}\sin\left(\theta+\dfrac{\pi}{4}\right)$

$\therefore\ -\sqrt{2}\leqq t\leqq\sqrt{2}$

Therefore, the maximum value is $3+\sqrt{2}$, at $t=-\sqrt{2}$ and
the minimum value is $\dfrac{3}{4}$, at $t=\dfrac{1}{2}$.

M179b

2. Let $y=a(\sin x+\cos x)+\sin 2x$. ($a$ is a positive constant.)

(1) Let $t=\sin x+\cos x$, and express y in terms of t.

[Sol] $t^2=\sin^2 x+2\sin x\cos x+\cos^2 x$

$\quad =1+\sin 2x$ $\quad\quad$ $\boxed{2\sin x\cos x=\sin 2x}$

$\quad \therefore\ \sin 2x=t^2-1$

$\quad \therefore\ y=t^2+at-1$

(2) Find the range of values that $t=\sin x+\cos x$ can take.

[Sol] $t=\sin x+\cos x=\sqrt{2}\left(\dfrac{\sqrt{2}}{2}\sin x+\dfrac{\sqrt{2}}{2}\cos x\right)$

$\quad =\sqrt{2}\left(\sin x\cos\dfrac{\pi}{4}+\cos x\sin\dfrac{\pi}{4}\right)=\sqrt{2}\sin\left(x+\dfrac{\pi}{4}\right)$

$\quad \therefore\ -\sqrt{2}\leqq t\leqq\sqrt{2}$

(3) Using a, express the maximum value M and the minimum value m of y respectively.

[Sol] $y=t^2+at-1$

$\quad =\left(t+\dfrac{a}{2}\right)^2-\dfrac{a^2}{4}-1$

The axis is $t=-\dfrac{a}{2}$. Since $a>0$, the axis<0

(i) When $-\sqrt{2}\leqq-\dfrac{a}{2}<0$,

i.e. $0<a\leqq 2\sqrt{2}$, $\quad\quad$ $\boxed{-\sqrt{2}\leqq\text{the axis}<0}$

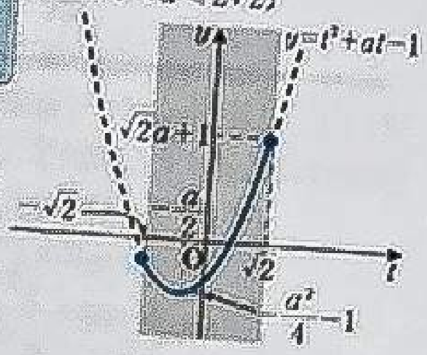
the maximum value is $M=\sqrt{2}a+1$,

at $t=\sqrt{2}$ and

the minimum value is $m=-\dfrac{a^2}{4}-1$,

at $t=-\dfrac{a}{2}$.

(When $0<a\leqq 2\sqrt{2}$)



(ii) When $-\dfrac{a}{2}<-\sqrt{2}$,

i.e. $2\sqrt{2}<a$, $\quad\quad$ $\boxed{\text{the axis}<-\sqrt{2}}$

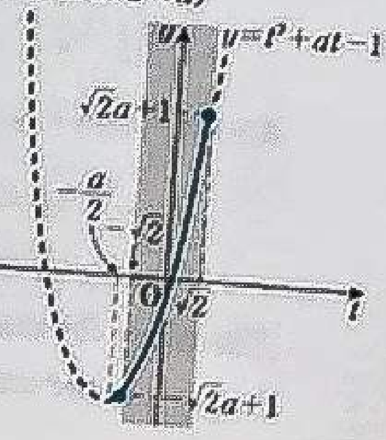
the maximum value is $M=\sqrt{2}a+1$,

at $t=\sqrt{2}$ and

the minimum value is $m=-\sqrt{2}a+1$,

at $t=-\sqrt{2}$.

(When $2\sqrt{2}<a$)



Addition Formulas 3

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1. Given $0 \leqq \theta < 2\pi$, solve the equation $\sin\theta + \sqrt{3}\cos\theta = -\sqrt{2}$. ➡ M I 72

[Sol] $\sin\theta + \sqrt{3}\cos\theta = 2\left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right)$

$$= 2\left(\sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3}\right)$$

$$= 2\sin\left(\theta + \frac{\pi}{3}\right)$$

$\therefore\ 2\sin\left(\theta + \frac{\pi}{3}\right) = -\sqrt{2}$

So, $\sin\left(\theta + \frac{\pi}{3}\right) = -\frac{\sqrt{2}}{2}$

Since $0 \leqq \theta < 2\pi$, $\frac{\pi}{3} \leqq \theta + \frac{\pi}{3} < \frac{7}{3}\pi$

$\theta + \frac{\pi}{3} = \frac{5}{4}\pi,\ \frac{7}{4}\pi$

$\therefore\ \theta = \frac{11}{12}\pi,\ \frac{17}{12}\pi$

2. Find the maximum and minimum values of function $y = \sqrt{3}\sin\theta - \cos\theta$ $(0 \leqq \theta < 2\pi)$, and state the corresponding values of θ. ➡ M I 74

[Sol] $y = 2\left(\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right)$

$$= 2\left[\sin\theta\cos\left(-\frac{\pi}{6}\right) + \cos\theta\sin\left(-\frac{\pi}{6}\right)\right]$$

$$= 2\sin\left(\theta - \frac{\pi}{6}\right)$$

Since $0 \leqq \theta < 2\pi$, $-\frac{\pi}{6} \leqq \theta - \frac{\pi}{6} < \frac{11}{6}\pi$; therefore, $-1 \leqq \sin\left(\theta - \frac{\pi}{6}\right) \leqq 1$

$\therefore\ -2 \leqq y \leqq 2$

Also, when $\sin\left(\theta - \frac{\pi}{6}\right) = 1$, $\theta - \frac{\pi}{6} = \frac{\pi}{2}$, i.e. $\theta = \frac{2}{3}\pi$

when $\sin\left(\theta - \frac{\pi}{6}\right) = -1$, $\theta - \frac{\pi}{6} = \frac{3}{2}\pi$, i.e. $\theta = \frac{5}{3}\pi$

Therefore, the maximum value is 2, at $\theta = \frac{2}{3}\pi$ and

the minimum value is -2, at $\theta = \frac{5}{3}\pi$.

3. Evaluate the following expression. ➡ MI76

$$\cos 20^\circ \cos 40^\circ \cos 80^\circ$$
$$=\frac{1}{2}[\cos(20^\circ+40^\circ)+\cos(20^\circ-40^\circ)]\cos 80^\circ$$
$$=\frac{1}{2}[\cos 60^\circ+\cos(-20^\circ)]\cos 80^\circ$$
$$=\frac{1}{2}\left(\frac{1}{2}+\cos 20^\circ\right)\cos 80^\circ$$
$$=\frac{1}{4}\cos 80^\circ+\frac{1}{2}\cos 20^\circ\cos 80^\circ$$
$$=\frac{1}{4}\cos 80^\circ+\frac{1}{2}\cdot\frac{1}{2}[\cos(20^\circ+80^\circ)+\cos(20^\circ-80^\circ)]$$
$$=\frac{1}{4}\cos 80^\circ+\frac{1}{4}[\cos 100^\circ+\cos(-60^\circ)]$$
$$=\frac{1}{4}\cos 80^\circ+\frac{1}{4}\cos 100^\circ+\frac{1}{4}\cdot\frac{1}{2}$$
$$=\frac{1}{4}\cos 80^\circ-\frac{1}{4}\cos 80^\circ+\frac{1}{8}$$
$$=\frac{1}{8}$$

4. Given $0\leqslant\theta\leqslant\frac{\pi}{2}$, solve the equation $\cos 7\theta+\cos 3\theta-\cos 2\theta=0$. ➡ MI78

[Sol] $2\cos\frac{7\theta+3\theta}{2}\cos\frac{7\theta-3\theta}{2}-\cos 2\theta=0$

$2\cos 5\theta\cos 2\theta-\cos 2\theta=0$

$\cos 2\theta(2\cos 5\theta-1)=0$

$\cos 2\theta=0,\ \cos 5\theta=\frac{1}{2}$

When $\cos 2\theta=0$,

since $0\leqslant\theta\leqslant\frac{\pi}{2}$, $0\leqslant 2\theta\leqslant\pi$

$\therefore\ 2\theta=\frac{\pi}{2}\quad\therefore\ \theta=\frac{\pi}{4}$

Also, when $\cos 5\theta=\frac{1}{2}$,

since $0\leqslant\theta\leqslant\frac{\pi}{2}$, $0\leqslant 5\theta\leqslant\frac{5}{2}\pi$

$\therefore\ 5\theta=\frac{\pi}{3},\ \frac{5}{3}\pi,\ \frac{7}{3}\pi\quad\therefore\ \theta=\frac{\pi}{15},\ \frac{\pi}{3},\ \frac{7}{15}\pi$

$\therefore\ \theta=\frac{\pi}{15},\ \frac{\pi}{4},\ \frac{\pi}{3},\ \frac{7}{15}\pi$

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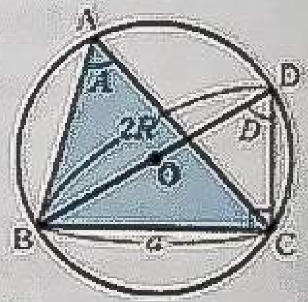
(1) As shown in the diagram below, given $\triangle$ABC with an acute angle A, prove that $\dfrac{a}{\sin A}=2R$, where R is the radius of the circumscribed circle of $\triangle$ABC and BC$=a$.



[Sol] Draw the diameter BD from point B.

Then, $A=\boxed{D}$ ⟵ Inscribed Angle Theorem (※1)

$\angle$BCD$=90°$ ⟵ Diameter and Inscribed Angle (※2)

$\therefore\ \sin A=\sin\boxed{D}=\boxed{\dfrac{a}{2R}}$

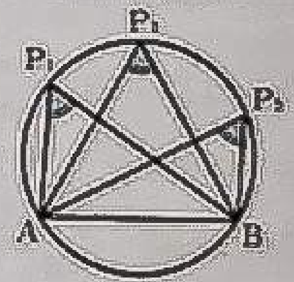
$\therefore\ \dfrac{a}{\sin A}=2R$

Answers: D, D, $\dfrac{a}{2R}$

An angle $0°<\theta<90°$ is called an ***acute angle***, and an angle $90°<\theta<180°$ is called an ***obtuse angle***.

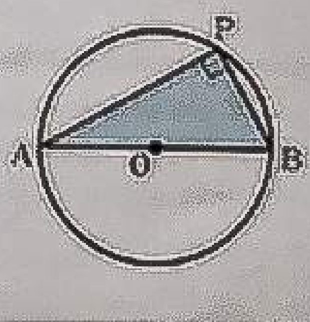
(※1) *Inscribed Angle Theorem*

Given that points P_1, P_2, P_3, ... lie on the same part of the circumference of a circle with respect to line segment AB, the angles $\angle AP_1B$, $\angle AP_2B$, $\angle AP_3B$, ... are all equal.



(※2) *Diameter and Inscribed Angle*

Place point P on the circumference of a circle where line segment AB is a diameter, then $\angle APB=90°$.



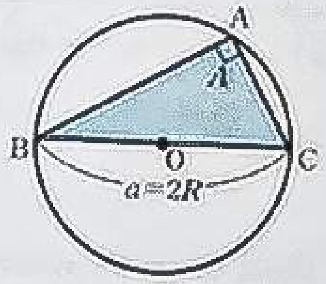
〔II〕 As shown in the diagram below, given $\triangle ABC$ with a right angle A, prove that $\dfrac{a}{\sin A}=2R$, where R is the radius of the circumscribed circle of $\triangle ABC$ and $BC=a$.

[Sol] Since $A=90°$, BC is the diameter of the circle.

Also, $\sin A=\boxed{1}$

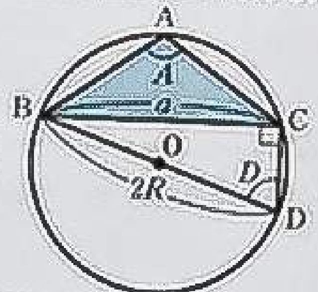
$\therefore\ a=2R=2R\boxed{\sin A}$

$\therefore\ \dfrac{a}{\sin A}=2R$



Answers: 1, sinA

〔III〕 As shown in the diagram below, given $\triangle ABC$ with an obtuse angle A, prove that $\dfrac{a}{\sin A}=2R$, where R is the radius of the circumscribed circle of $\triangle ABC$ and $BC=a$.



[Sol] Draw the diameter BD from point B. ← Diameter and Inscribed Angle

Then, $\angle DCB=\boxed{90°}$

Since quadrilateral ABDC is circumscribed in the circle,

$A+D=\boxed{180°}$ ← Quadrilateral Inscribed in the Circle (※3)

$\therefore\ a=BD\sin\boxed{D}=2R\sin(\boxed{180°-A})$

$=2R\boxed{\sin A}$ ← $\sin(180°-\theta)=\sin\theta$

$\therefore\ \dfrac{a}{\sin A}=2R$

Answers: 90°, 180°, D, 180°−A, sinA

From (I)~(III), $\dfrac{a}{\sin A}=2R$ is true for any triangle.

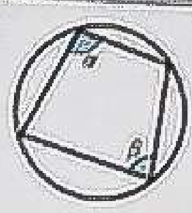
Similarly, $\dfrac{b}{\sin B}=2R$ and $\dfrac{c}{\sin C}=2R$ are also true.

Therefore, $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R$. This is called the ***Law of Sines.***

(※3) ***Quadrilateral Inscribed in the Circle***

Given that a quadrilateral is inscribed in a circle, the sum of the opposite angles is 180°.

$\alpha+\beta=180°$



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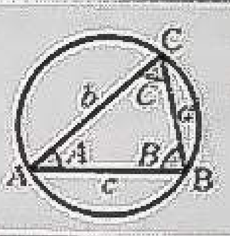
Law of Sines

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$

(R is the radius of the circumscribed circle of $\triangle$ABC)



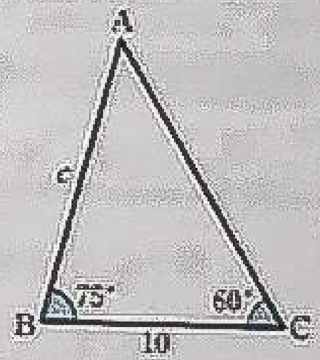
Ex. Given $\triangle$ABC where $a=10$, $B=75°$ and $C=60°$, find c.

[Sol] $A=180°-(75°+60°)=45°$

$$\frac{10}{\sin 45°}=\frac{c}{\sin 60°} \longleftarrow \boxed{\frac{a}{\sin A}=\frac{c}{\sin C}}$$

$$\therefore\ c=\frac{10\sin 60°}{\sin 45°}$$

$$=5\sqrt{6}$$



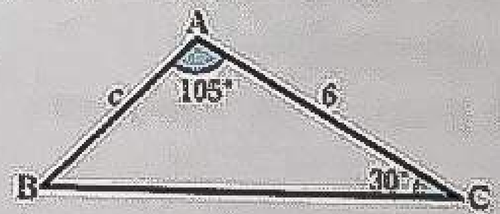
1. Given $\triangle$ABC where $b=6$, $A=105°$ and $C=30°$, find c.

[Sol] $B=180°-(105°+30°)=45°$

$$\frac{6}{\sin 45°}=\frac{c}{\sin 30°}$$

$$\therefore\ c=\frac{6\sin 30°}{\sin 45°}$$

$$=3\sqrt{2}$$



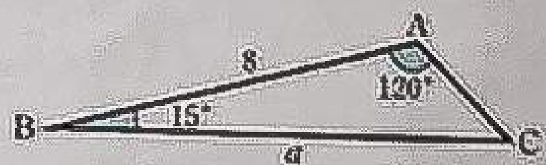
2. Given $\triangle$ABC where $c=8$, $A=120°$ and $B=15°$, find a.

[Sol] $C=180°-(120°+15°)=45°$

$$\frac{a}{\sin 120°}=\frac{8}{\sin 45°}$$

$$\therefore\ a=\frac{8\sin 120°}{\sin 45°}$$

$$=4\sqrt{6}$$



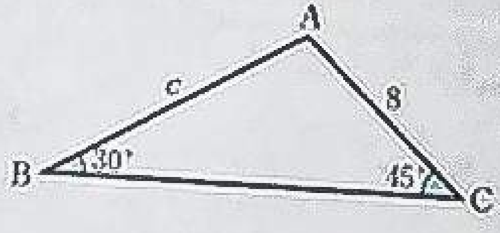
M182b

3. Given $\triangle ABC$ where $b=8$, $B=30°$ and $C=45°$, find c.

[Sol] $\dfrac{8}{\sin 30°}=\dfrac{c}{\sin 45°}$

$\therefore\ c=\dfrac{8\sin 45°}{\sin 30°}$

$=8\sqrt{2}$

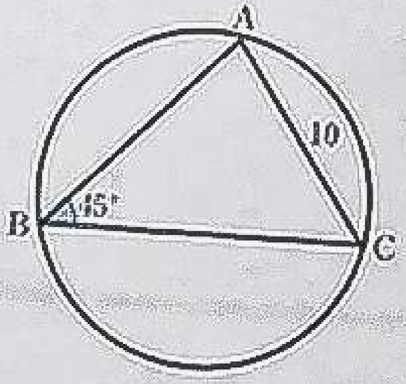


4. Given $\triangle ABC$ where $b=10$ and $B=45°$, find the radius R of the circumscribed circle.

[Sol] $\dfrac{10}{\sin 45°}=2R \Longleftarrow \boxed{\dfrac{b}{\sin B}=2R}$

$\therefore\ R=\dfrac{10}{2\sin 45°}$

$=5\sqrt{2}$



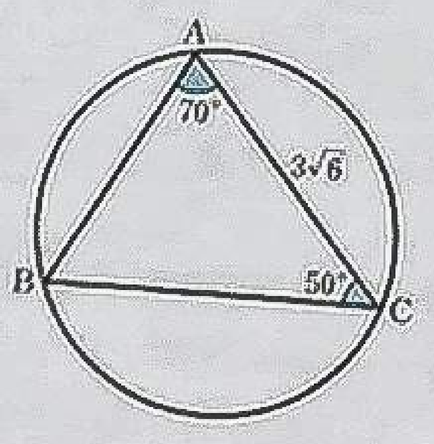
5. Given $\triangle ABC$ where $b=3\sqrt{6}$, $A=70°$ and $C=50°$, find the radius R of the circumscribed circle.

[Sol] $B=180°-(70°+50°)=60°$

$\dfrac{3\sqrt{6}}{\sin 60°}=2R$

$\therefore\ R=\dfrac{3\sqrt{6}}{2\sin 60°}$

$=3\sqrt{2}$



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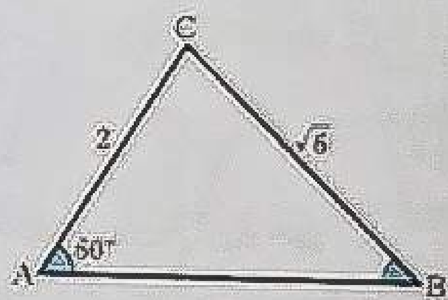
Ex. Given $\triangle ABC$ where $a=\sqrt{6}$, $b=2$ and $A=60°$, find B.

[Sol] $\dfrac{\sqrt{6}}{\sin 60°}=\dfrac{2}{\sin B}$

$\therefore\ \sin B=\dfrac{2\sin 60°}{\sqrt{6}}$

$=\dfrac{\sqrt{2}}{2}$



Since $A=60°$, $0°<B<120°$; therefore, ⟵ $A+B+C=180°$

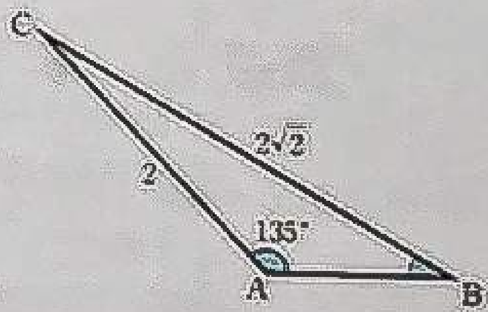
$B=45°$ ⟵ $B=135°$ cannot be true.

1. Given $\triangle ABC$ where $a=2\sqrt{2}$, $b=2$ and $A=135°$, find B.

[Sol] $\dfrac{2\sqrt{2}}{\sin 135°}=\dfrac{2}{\sin B}$

$\therefore\ \sin B=\dfrac{2\sin 135°}{2\sqrt{2}}$

$=\dfrac{1}{2}$



Since $A=135°$, $0°<B<45°$; therefore,

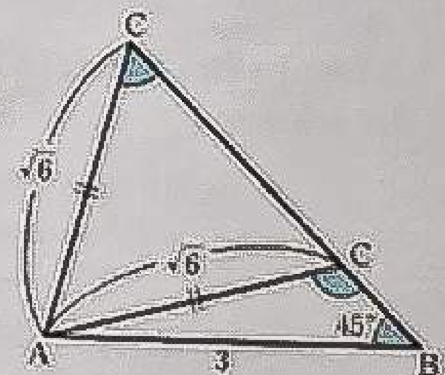
$B=30°$

2. Given $\triangle ABC$ where $b=\sqrt{6}$, $c=3$ and $B=45°$, find C.

[Sol] $\dfrac{\sqrt{6}}{\sin 45°}=\dfrac{3}{\sin C}$

$\therefore\ \sin C=\dfrac{3\sin 45°}{\sqrt{6}}$

$=\dfrac{\sqrt{3}}{2}$



Since $B=45°$, $0°<C<135°$; therefore,

$C=60°,\ 120°$ ⟵ Both are true.

M183b

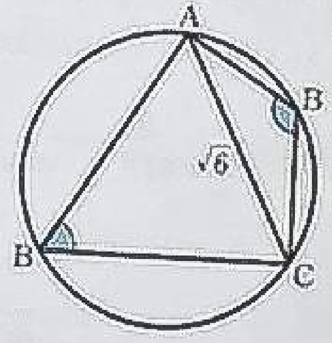
3. Given $\triangle ABC$ where $b=\sqrt{6}$ and the radius of the circumscribed circle is $R=\sqrt{2}$, find B.

[Sol] $\dfrac{\sqrt{6}}{\sin B}=2\cdot\sqrt{2}$

$\therefore\ \sin B=\dfrac{\sqrt{6}}{2\sqrt{2}}$

$=\dfrac{\sqrt{3}}{2}$

$\therefore\ B=60°,\ 120°$ ⟵ $0°<B<180°$



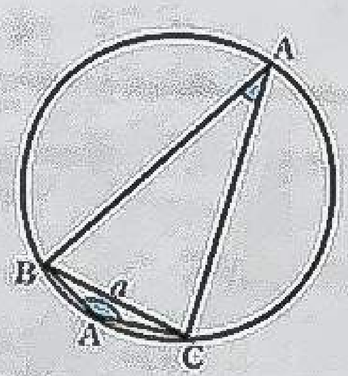
4. Given $\triangle ABC$ where a is equal to the radius R of the circumscribed circle ($a=R$), find A.

[Sol] $\dfrac{a}{\sin A}=2\cdot a$

$\therefore\ \sin A=\dfrac{a}{2a}$

$=\dfrac{1}{2}$

$\therefore\ A=30°,\ 150°$ ⟵ $0°<A<180°$



5. Given $\triangle ABC$ where $a=3\sqrt{3}$, $c=3\sqrt{2}$ and $A=60°$, find B.

[Sol] $\dfrac{3\sqrt{3}}{\sin 60°}=\dfrac{3\sqrt{2}}{\sin C}$

$\therefore\ \sin C=\dfrac{3\sqrt{2}\sin 60°}{3\sqrt{3}}$

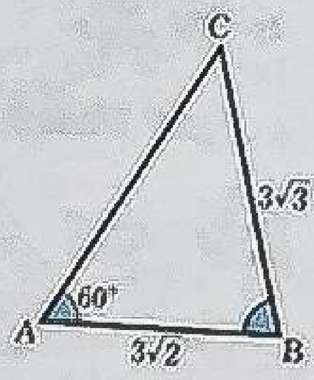
$=\dfrac{\sqrt{2}}{2}$

Since $A=60°$, $0°<C<120°$; therefore,

$C=45°$

$\therefore\ B=180°-(60°+45°)$

$=75°$



M184a

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As shown in the diagram below, when A and B are both acute angles, prove th
$a^2=b^2+c^2-2bc\cos A$.

[Sol] In $\triangle$ABC, drop a perpendicular CH from vertex C to AB. In $\triangle$ACH,

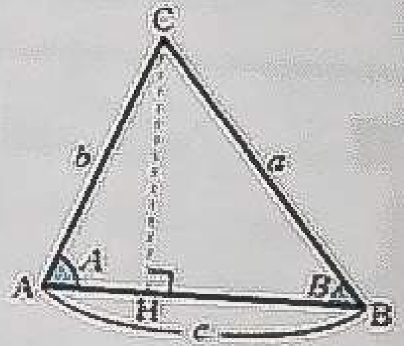
$\mathrm{CH}=\boxed{b\sin A}$, $\mathrm{AH}=\boxed{b\cos A}$

Also, $\mathrm{BH}=\mathrm{AB}-\mathrm{AH}=\boxed{c-b\cos A}$

In $\triangle$BCH,

$$a^2=\mathrm{CH}^2+\mathrm{BH}^2$$
$$=(\boxed{b\sin A})^2+(\boxed{c-b\cos A})^2$$
$$=b^2+c^2-2bc\cos A$$

$\boxed{\sin^2 A+\cos^2 A=1}$



Therefore, $a^2=b^2+c^2-2bc\cos A$ is true.

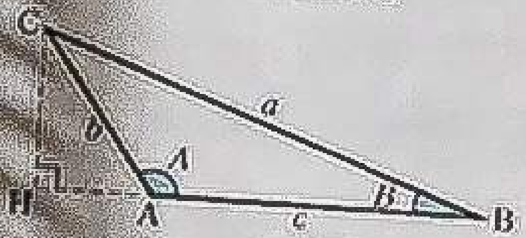
Answers: $b\sin A$, $b\cos A$, $c-b\cos A$, $b\sin A$, $c-b\cos A$

When A is an obtuse angle,

$\mathrm{CH}=b\sin(180°-A)=b\sin A$

$\mathrm{BH}=\mathrm{AB}+\mathrm{AH}$
$=c+b\cos(180°-A)$
$=c-b\cos A$

$a^2=\mathrm{CH}^2+\mathrm{BH}^2$
$=(b\sin A)^2+(c-b\cos A)^2$
$=b^2+c^2-2bc\cos A$

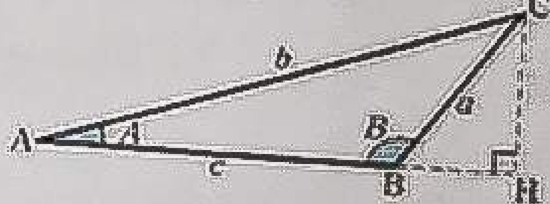


When B is an obtuse angle,

$\mathrm{CH}=b\sin A$

$\mathrm{BH}=\mathrm{AH}-\mathrm{AB}$
$=b\cos A-c$

$a^2=\mathrm{CH}^2+\mathrm{BH}^2$
$=(b\sin A)^2+(b\cos A-c)^2$
$=b^2+c^2-2bc\cos A$



Therefore, when either A or B is an obtuse angle, $a^2=b^2+c^2-2bc\cos A$.
It is also true when either A or B is a right angle.
Similarly, $b^2=c^2+a^2-2ca\cos B$ and $c^2=a^2+b^2-2ab\cos C$ are true.
This is called the ***Law of Cosines***.

M184b

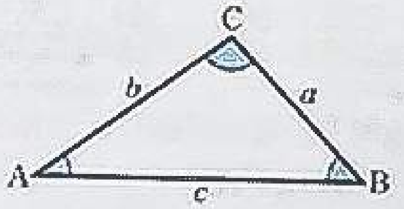
Law of Cosines

Given $\triangle ABC$,

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = c^2 + a^2 - 2ca\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$



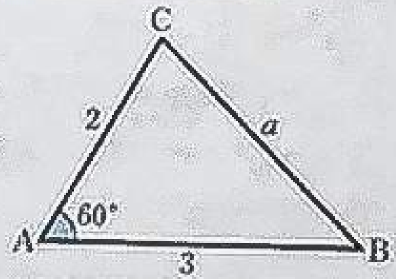
Ex. Given $\triangle ABC$ where $b=2$, $c=3$ and $A=60°$, find a.

[Sol] $a^2 = 2^2 + 3^2 - 2\cdot 2\cdot 3\cos 60°$ ⟵ $\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$

$= 7$

$a = \pm\sqrt{7}$

Since $a > 0$, $a = \sqrt{7}$



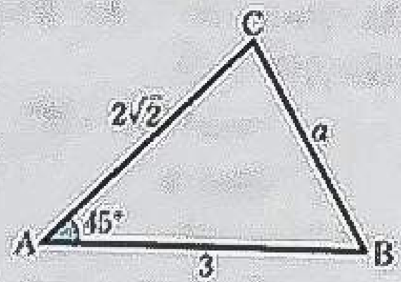
1. Given $\triangle ABC$ where $b = 2\sqrt{2}$, $c = 3$ and $A = 45°$, find a.

[Sol] $a^2 = (2\sqrt{2})^2 + 3^2 - 2\cdot 2\sqrt{2}\cdot 3\cos 45°$

$= 5$

$a = \pm\sqrt{5}$

Since $a > 0$, $a = \sqrt{5}$



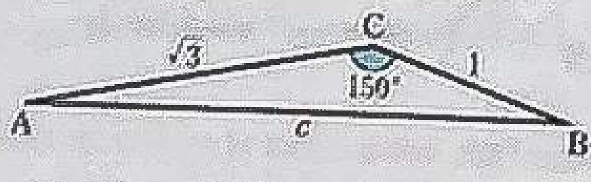
2. Given $\triangle ABC$ where $a = 1$, $b = \sqrt{3}$ and $C = 150°$, find c.

[Sol] $c^2 = 1^2 + (\sqrt{3})^2 - 2\cdot 1\cdot\sqrt{3}\cos 150°$ ⟵ $\boxed{c^2 = a^2 + b^2 - 2ab\cos C}$

$= 7$

$c = \pm\sqrt{7}$

Since $c > 0$, $c = \sqrt{7}$



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Ex. Given $\triangle ABC$ where $b=\sqrt{5}$, $c=\sqrt{2}$ and $B=45°$, find a.

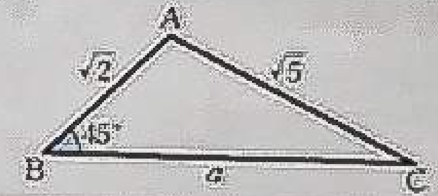
[Sol] $(\sqrt{5})^2=(\sqrt{2})^2+a^2-2\cdot\sqrt{2}\cdot a\cos 45°$ ⟵ $\boxed{b^2=c^2+a^2-2ca\cos B}$

$\therefore\ a^2-2a-3=0$

$(a+1)(a-3)=0$

$\therefore\ a=-1,\ 3$

Since $a>0$, $a=3$



1. Given $\triangle ABC$ where $b=\sqrt{7}$, $c=1$ and $B=120°$, find a.

[Sol] $(\sqrt{7})^2=1^2+a^2-2\cdot 1\cdot a\cos 120°$

$\therefore\ a^2+a-6=0$

$(a+3)(a-2)=0$

$\therefore\ a=-3,\ 2$

Since $a>0$, $a=2$

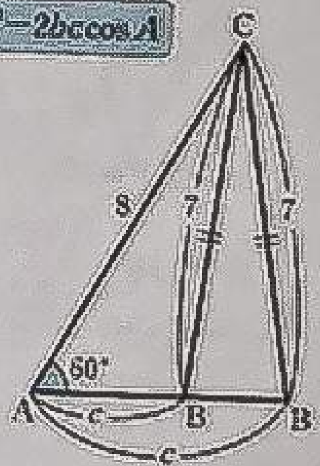
2. Given $\triangle ABC$ where $a=7$, $b=8$ and $A=60°$, find c.

[Sol] $7^2=8^2+c^2-2\cdot 8\cdot c\cos 60°$ ⟵ $\boxed{a^2=b^2+c^2-2bc\cos A}$

$\therefore\ c^2-8c+15=0$

$(c-3)(c-5)=0$

$\therefore\ c=3,\ 5$



M185b

The size of the angle can be found from the three sides by rearranging the Law of Cosines as follows:

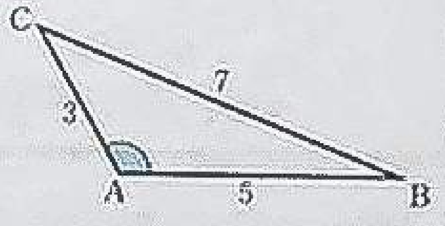
$$\cos A=\frac{b^2+c^2-a^2}{2bc},\quad \cos B=\frac{c^2+a^2-b^2}{2ca},\quad \cos C=\frac{a^2+b^2-c^2}{2ab}$$

3. Given $\triangle ABC$ where $a=7$, $b=3$ and $c=5$, find A.

[Sol] $\cos A=\frac{3^2+5^2-7^2}{2\cdot 3\cdot 5}$ $\quad \boxed{\cos A=\frac{b^2+c^2-a^2}{2bc}}$

$=-\frac{1}{2}$

$\therefore A=120^\circ$

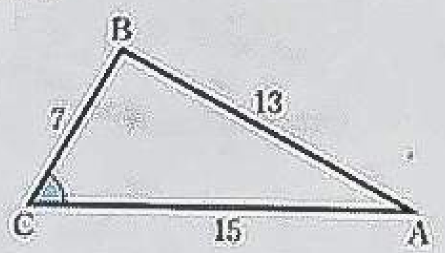


4. Given $\triangle ABC$ where $a=7$, $b=15$ and $c=13$, find C.

[Sol] $\cos C=\frac{7^2+15^2-13^2}{2\cdot 7\cdot 15}$ $\quad \boxed{\cos C=\frac{a^2+b^2-c^2}{2ab}}$

$=\frac{1}{2}$

$\therefore C=60^\circ$

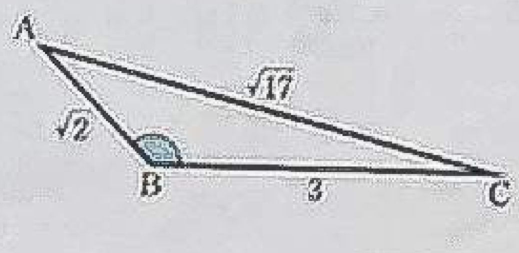


5. Given $\triangle ABC$ where $a=3$, $b=\sqrt{17}$ and $c=\sqrt{2}$, find B.

[Sol] $\cos B=\frac{(\sqrt{2})^2+3^2-(\sqrt{17})^2}{2\cdot\sqrt{2}\cdot 3}$ $\quad \boxed{\cos B=\frac{c^2+a^2-b^2}{2ca}}$

$=-\frac{\sqrt{2}}{2}$

$\therefore B=135^\circ$



M186a KUMON M186

Laws of Sines and Cosines

Name

Date / /

Time : to :

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Ex. Given $\triangle ABC$ where $a=\sqrt{2}$, $c=\sqrt{3}+1$ and $B=45°$, find b, A and C.

[Sol] $b^2=(\sqrt{3}+1)^2+(\sqrt{2})^2-2\cdot(\sqrt{3}+1)\cdot\sqrt{2}\cos 45°$ ← $b^2=c^2+a^2-2ca\cos B$

$=4$

Since $b>0$, $b=2$

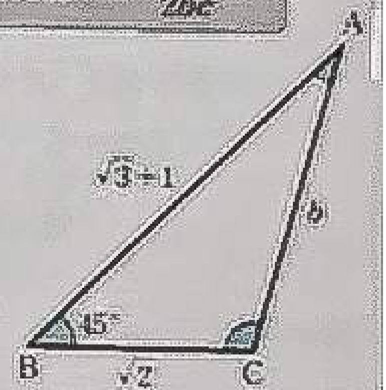
$\cos A=\dfrac{2^2+(\sqrt{3}+1)^2-(\sqrt{2})^2}{2\cdot 2\cdot(\sqrt{3}+1)}$ ← $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$

$=\dfrac{\sqrt{3}}{2}$

$\therefore A=30°$

$\therefore C=180°-(30°+45°)$

$=105°$



1. Given $\triangle ABC$ where $a=2$, $c=\sqrt{3}+1$ and $B=60°$, find b, A and C.

[Sol] $b^2=(\sqrt{3}+1)^2+2^2-2\cdot(\sqrt{3}+1)\cdot 2\cos 60°$

$=6$

Since $b>0$, $b=\sqrt{6}$

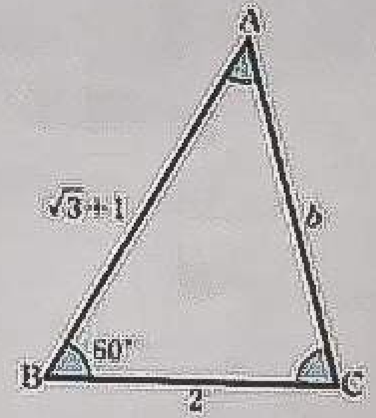
$\cos A=\dfrac{(\sqrt{6})^2+(\sqrt{3}+1)^2-2^2}{2\cdot\sqrt{6}\cdot(\sqrt{3}+1)}$

$=\dfrac{\sqrt{2}}{2}$

$\therefore A=45°$

$\therefore C=180°-(45°+60°)$

$=75°$



M186b

2. Given $\triangle ABC$ where $a=\sqrt{6}$, $b=\sqrt{3}-1$ and $C=45°$, find c, A and B.

[Sol] $c^2=(\sqrt{6})^2+(\sqrt{3}-1)^2-2\cdot\sqrt{6}\cdot(\sqrt{3}-1)\cos 45°$ ⟵ $\boxed{c^2=a^2+b^2-2ab\cos C}$

$=4$

Since $c>0$, $c=2$

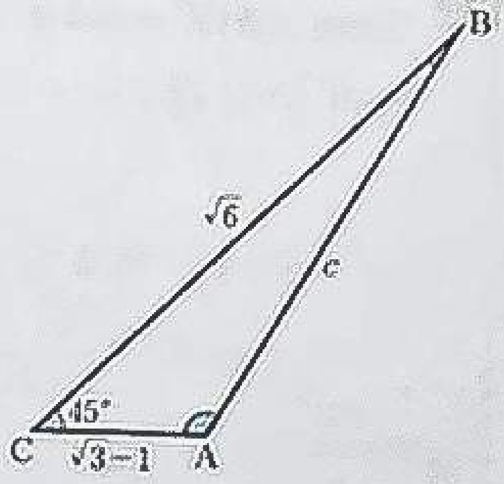
$$\cos A=\frac{(\sqrt{3}-1)^2+2^2-(\sqrt{6})^2}{2\cdot(\sqrt{3}-1)\cdot 2}$$

$$=-\frac{1}{2}$$

$\therefore\ A=120°$

$\therefore\ B=180°-(120°+45°)$

$=15°$



3. Given $\triangle ABC$ where $b=\sqrt{2}$, $c=\sqrt{3}-1$ and $A=135°$, find a, B and C.

[Sol] $a^2=(\sqrt{2})^2+(\sqrt{3}-1)^2-2\cdot\sqrt{2}\cdot(\sqrt{3}-1)\cos 135°$ ⟵ $\boxed{a^2=b^2+c^2-2bc\cos A}$

$=4$

Since $a>0$, $a=2$

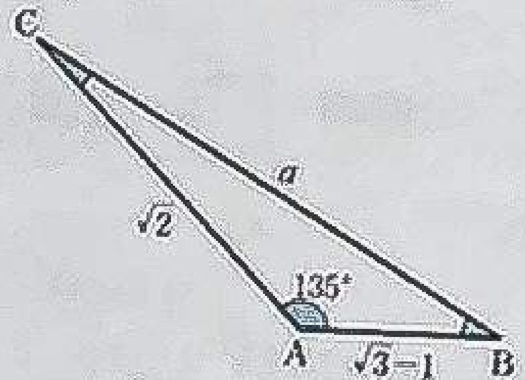
$$\cos B=\frac{(\sqrt{3}-1)^2+2^2-(\sqrt{2})^2}{2\cdot(\sqrt{3}-1)\cdot 2} \longleftarrow \boxed{\cos B=\frac{c^2+a^2-b^2}{2ca}}$$

$$=\frac{\sqrt{3}}{2}$$

$\therefore\ B=30°$

$\therefore\ C=180°-(135°+30°)$

$=15°$



Laws of Sines and Cosines

Name

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Ex. Given $\triangle ABC$ where $a=1$, $b=\sqrt{3}$ and $A=30°$, find c, B and C.

[Sol] $1^2=(\sqrt{3})^2+c^2-2\cdot\sqrt{3}\cdot c\cos 30°$ ← $a^2=b^2+c^2-2bc\cos A$

$\therefore\ c^2-3c+2=0$

$(c-1)(c-2)=0 \quad \therefore\ c=1,\ 2$

(i) When $c=1$,

$$\cos B=\frac{1^2+1^2-(\sqrt{3})^2}{2\cdot1\cdot1}=-\frac{1}{2} \quad \therefore\ B=120°$$

$\therefore\ C=180°-(30°+120°)=30°$

(ii) When $c=2$,

$$\cos B=\frac{2^2+1^2-(\sqrt{3})^2}{2\cdot2\cdot1}=\frac{1}{2} \quad \therefore\ B=60°$$

$\therefore\ C=180°-(30°+60°)=90°$

From (i) and (ii),

$c=1$, $B=120°$, $C=30°$ or $c=2$, $B=60°$, $C=90°$

1. Given $\triangle ABC$ where $b=3\sqrt{3}$, $c=3$ and $C=30°$, find a, A and B.

[Sol] $3^2=a^2+(3\sqrt{3})^2-2\cdot a\cdot 3\sqrt{3}\cos 30°$ ← $c^2=a^2+b^2-2ab\cos C$

$\therefore\ a^2-9a+18=0$

$(a-3)(a-6)=0 \quad \therefore\ a=3,\ 6$

(i) When $a=3$,

$$\cos A=\frac{(3\sqrt{3})^2+3^2-3^2}{2\cdot3\sqrt{3}\cdot3}=\frac{\sqrt{3}}{2} \quad \therefore\ A=30°$$

$\therefore\ B=180°-(30°+30°)=120°$

(ii) When $a=6$,

$$\cos A=\frac{(3\sqrt{3})^2+3^2-6^2}{2\cdot3\sqrt{3}\cdot3}=0 \quad \therefore\ A=90°$$

$\therefore\ B=180°-(90°+30°)=60°$

From (i) and (ii),

$a=3$, $A=30°$, $B=120°$ or $a=6$, $A=90°$, $B=60°$

2. Given $\triangle ABC$ where $a=\sqrt{6}$, $b=2$ and $B=45°$, find c, A and C.

[Sol] $2^2=c^2+(\sqrt{6})^2-2\cdot c\cdot\sqrt{6}\cos 45°$ ← $b^2=c^2+a^2-2ca\cos B$

$\therefore\ c^2-2\sqrt{3}c+2=0$

Quadratic Formula II(J102)
$c=\sqrt{3}\pm\sqrt{(-\sqrt{3})^2-1\cdot 2}$

$\therefore\ c=\sqrt{3}\pm 1$

(i) When $c=\sqrt{3}+1$,

$$\cos A=\frac{2^2+(\sqrt{3}+1)^2-(\sqrt{6})^2}{2\cdot 2\cdot(\sqrt{3}+1)}=\frac{1}{2}\quad \therefore\ A=60°$$

$\therefore\ C=180°-(60°+45°)=75°$

(ii) When $c=\sqrt{3}-1$,

$$\cos A=\frac{2^2+(\sqrt{3}-1)^2-(\sqrt{6})^2}{2\cdot 2\cdot(\sqrt{3}-1)}=-\frac{1}{2}\quad \therefore\ A=120°$$

$\therefore\ C=180°-(120°+45°)=15°$

From (i) and (ii),

$c=\sqrt{3}+1$, $A=60°$, $C=75°$ or $c=\sqrt{3}-1$, $A=120°$, $C=15°$

3. Given $\triangle ABC$ where $a=1+\sqrt{3}$, $b=2$ and $B=45°$, find c, A and C.

[Sol] $2^2=c^2+(1+\sqrt{3})^2-2\cdot c\cdot(1+\sqrt{3})\cos 45°$

$\therefore\ c^2-(\sqrt{2}+\sqrt{6})c+2\sqrt{3}=0$

$(c-\sqrt{2})(c-\sqrt{6})=0\quad \therefore\ c=\sqrt{2},\ \sqrt{6}$

To find A first,
$\cos A=\frac{\sqrt{2}-\sqrt{6}}{4}$
Since it is difficult to find A, find C first.

(i) When $c=\sqrt{2}$,

$$\cos C=\frac{(1+\sqrt{3})^2+2^2-(\sqrt{2})^2}{2\cdot(1+\sqrt{3})\cdot 2}=\frac{\sqrt{3}}{2}\quad \therefore\ C=30°$$

$\therefore\ A=180°-(45°+30°)=105°$

(ii) When $c=\sqrt{6}$,

$$\cos C=\frac{(1+\sqrt{3})^2+2^2-(\sqrt{6})^2}{2\cdot(1+\sqrt{3})\cdot 2}=\frac{1}{2}\quad \therefore\ C=60°$$

$\therefore\ A=180°-(45°+60°)=75°$

From (i) and (ii),

$c=\sqrt{2}$, $A=105°$, $C=30°$ or $c=\sqrt{6}$, $A=75°$, $C=60°$

Laws of Sines and Cosines

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The Law of Sines $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$ can be rearranged as follows:

$a=2R\sin A,\ b=2R\sin B,\ c=2R\sin C$

Therefore,

$a:b:c=2R\sin A:2R\sin B:2R\sin C$

$=\sin A:\sin B:\sin C$

Using the identities above, solve the following questions.

Ex. Given $\triangle$ABC such that $\sin A:\sin B:\sin C=5:7:8$, find B.

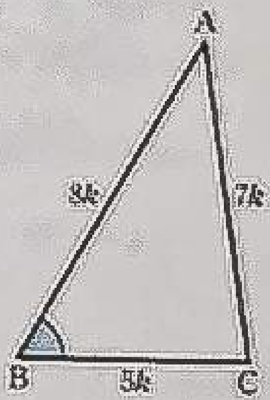
[Sol] $a:b:c=5:7:8$

Therefore, let k be a positive constant.

$a=5k,\ b=7k,\ c=8k$

$$\cos B=\frac{(8k)^2+(5k)^2-(7k)^2}{2\cdot 8k\cdot 5k}=\frac{1}{2}$$

$\therefore B=60°$



(1) Given $\triangle$ABC such that $\sin A:\sin B:\sin C=\sqrt{3}:2:(\sqrt{2}+1)$, find A.

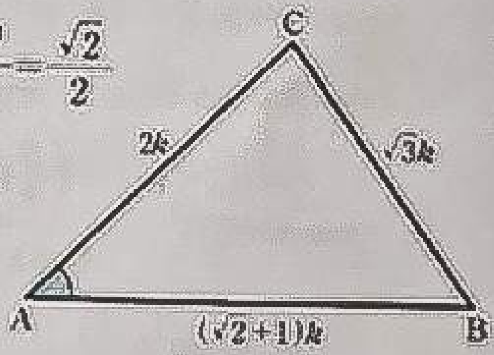
[Sol] $a:b:c=\sqrt{3}:2:(\sqrt{2}+1)$

Therefore, let k be a positive constant.

$a=\sqrt{3}k,\ b=2k,\ c=(\sqrt{2}+1)k$

$$\cos A=\frac{(2k)^2+\{(\sqrt{2}+1)k\}^2-(\sqrt{3}k)^2}{2\cdot 2k\cdot(\sqrt{2}+1)k}=\frac{\sqrt{2}}{2}$$

$\therefore A=45°$



M188b

The following statement is true for all triangles.

Relationships between Angles and Sides of a Triangle

The relationship between the lengths of two sides corresponds to the relationship between the sizes of the opposite angles.
(The angle facing the longest side is the largest angle.)

(2) Given $\triangle ABC$ such that $\sin A : \sin B : \sin C = 13 : 8 : 7$, find the size of the largest angle.

[Sol] $a : b : c = 13 : 8 : 7$

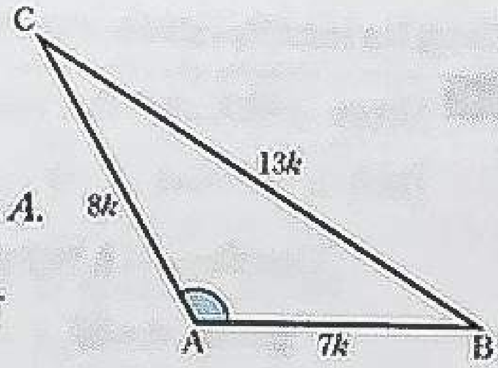
Therefore, let k be a positive constant.

$a = 13k$, $b = 8k$, $c = 7k$

Since a is the longest, the largest angle is A.

$$\cos A = \frac{(8k)^2 + (7k)^2 - (13k)^2}{2 \cdot 8k \cdot 7k} = -\frac{1}{2}$$

$\therefore\ A = 120°$



(3) Given $\triangle ABC$ such that $(b+c) : (c+a) : (a+b) = 4 : 5 : 6$, find the size of the largest angle.

[Sol] Let k be a positive constant.

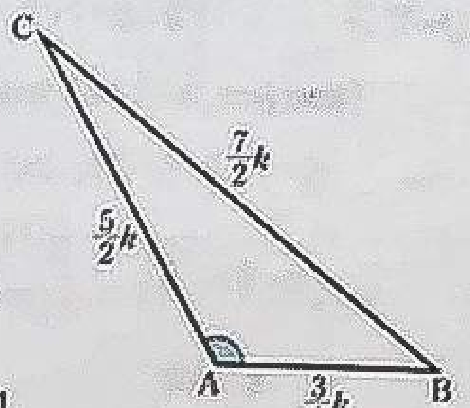
$$\begin{cases} b+c=4k & \cdots① \\ c+a=5k & \cdots② \\ a+b=6k & \cdots③ \end{cases}$$

From ①~③, $a = \frac{7}{2}k$, $b = \frac{5}{2}k$, $c = \frac{3}{2}k$

Since a is the longest, the largest angle is A.

$$\cos A = \frac{\left(\frac{5}{2}k\right)^2 + \left(\frac{3}{2}k\right)^2 - \left(\frac{7}{2}k\right)^2}{2 \cdot \frac{5}{2}k \cdot \frac{3}{2}k} = -\frac{1}{2}$$

$\therefore\ A = 120°$



Laws of Sines and Cosines

Name

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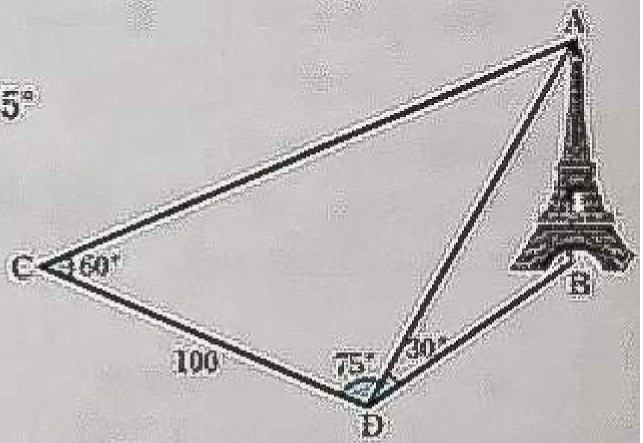
1. When looking at the top and the bottom of a tower, A and B, from two points C and D, $\angle ACD = 60°$, $\angle ADC = 75°$ and $\angle ADB = 30°$. Given that the distance between points C and D is 100, find the height AB of the tower.



[Sol] In $\triangle ACD$,

$$\angle CAD = 180° - (60° + 75°) = 45°$$

$$\frac{AD}{\sin 60°} = \frac{100}{\sin 45°}$$

$$\therefore \ AD = \frac{100 \sin 60°}{\sin 45°}$$

$$= 50\sqrt{6}$$

Since $\triangle ABD$ is a right-angled triangle where $\angle ABD = 90°$,

$$AB = AD \sin 30°$$

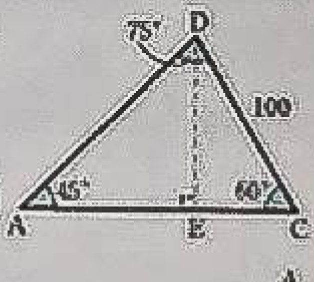
$$= 25\sqrt{6}$$

Alternative Solution

In $\triangle ACD$,

$\angle CAD = 180° - (60° + 75°) = 45°$

Drop a perpendicular from point D to side AC to place point E.



In $\angle CDE$,

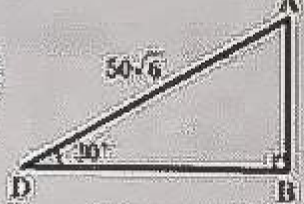
$100 : DE = 2 : \sqrt{3}$ ← Since $CD : DE = 2 : \sqrt{3}$

$\therefore \ DE = 50\sqrt{3}$

In $\triangle ADE$,

$AD : 50\sqrt{3} = \sqrt{2} : 1$ ← Since $AD : DE = \sqrt{2} : 1$

$\therefore \ AD = 50\sqrt{6}$



Also, in $\triangle ABD$,

$AB : 50\sqrt{6} = 1 : 2$ ← Since $AB : AD = 1 : 2$

$\therefore \ AB = 25\sqrt{6}$

M189b

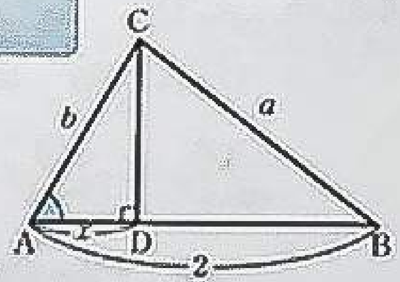
2. Let the length of each side of $\triangle$ABC with area 1 be AB=2, BC=a and CA=b. Given that CD is a perpendicular dropped from point C to side AB, solve the following questions.

(1) Given AD=x, express $a^2+(2\sqrt{3}-1)b^2$ in terms of x.

[Sol] In $\triangle$ABC,

since $\frac{1}{2}\cdot 2\cdot \text{CD}=1$, ⟵ Since the area of $\triangle$ABC is $\frac{1}{2}\cdot\text{AB}\cdot\text{CD}=1$

CD=1 ···①



In $\triangle$ACD, from ①,

$b^2=x^2+1$ ···② ⟵ $\text{AC}^2=\text{AD}^2+\text{CD}^2$

Also, $b\cos A=x$ ···③

In $\triangle$ABC,

$$a^2=b^2+2^2-2\cdot b\cdot 2\cos A$$
$$=b^2-4b\cos A+4 \quad \cdots ④$$

From ②~④,

$$a^2+(2\sqrt{3}-1)b^2=(b^2-4b\cos A+4)+(2\sqrt{3}-1)b^2 \quad \text{⟵ From ④}$$
$$=2\sqrt{3}b^2-4b\cos A+4$$
$$=2\sqrt{3}(x^2+1)-4x+4 \quad \text{⟵ From ② and ③}$$
$$=2\sqrt{3}x^2-4x+2\sqrt{3}+4$$

(2) Find the value of x which minimizes $a^2+(2\sqrt{3}-1)b^2$. Also, find the size of $\angle$BAC.

[Sol] From (1), $a^2+(2\sqrt{3}-1)b^2=2\sqrt{3}x^2-4x+2\sqrt{3}+4$

$$=2\sqrt{3}\left(x-\frac{\sqrt{3}}{3}\right)^2+\frac{4\sqrt{3}}{3}+4$$

Therefore, the value of x which minimizes $a^2+(2\sqrt{3}-1)b^2$ is $x=\frac{\sqrt{3}}{3}$.

Then, $\tan A=\frac{1}{x}=\sqrt{3}$ ⟵ From $\tan A=\frac{\text{CD}}{x}$ and ①

$\therefore$ $\angle\text{BAC}=60°$ ⟵ From $\angle\text{BAC}=A$

Laws of Sines and Cosines

Name

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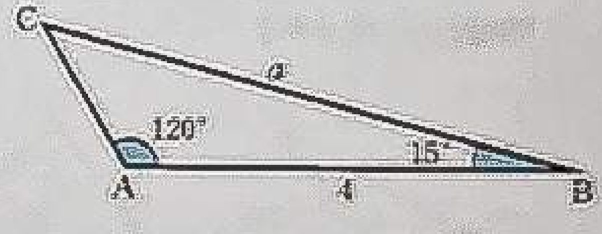
1. Given $\triangle ABC$ where $c=4$, $A=120°$ and $B=15°$, find a. ➡ M I 87

[Sol] $C=180°-(120°+15°)=45°$

$$\frac{a}{\sin 120°}=\frac{4}{\sin 45°}$$

$$\therefore\ a=\frac{4\sin 120°}{\sin 45°}$$

$$=2\sqrt{6}$$



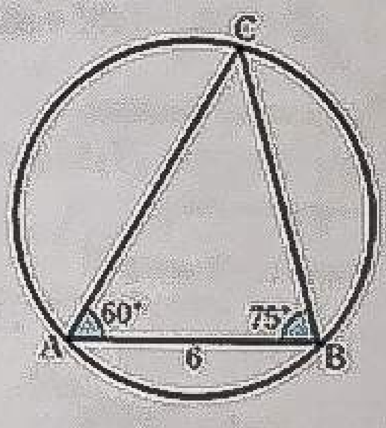
2. Given $\triangle ABC$ where $c=6$, $A=60°$ and $B=75°$, find the radius R of the circumscribed circle. ➡ M I 82

[Sol] $C=180°-(60°+75°)=45°$

$$\frac{6}{\sin 45°}=2R$$

$$\therefore\ R=\frac{6}{2\sin 45°}$$

$$=3\sqrt{2}$$



3. Given $\triangle ABC$ where $a=\sqrt{7}$, $b=\sqrt{3}$ and $A=150°$, find c. ➡ M185

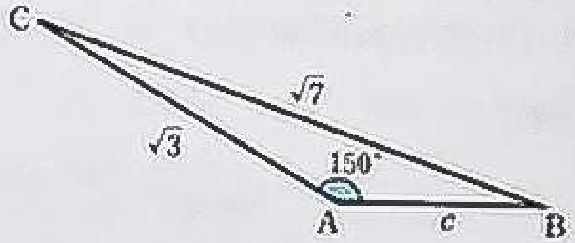
[Sol] $(\sqrt{7})^2=(\sqrt{3})^2+c^2-2\cdot\sqrt{3}\cdot c\cos 150°$

$\therefore c^2+3c-4=0$

$(c+4)(c-1)=0$

$\therefore c=-4,\ 1$

Since $c>0$, $c=1$



4. Given $\triangle ABC$ where $b=2$, $c=\sqrt{6}+\sqrt{2}$ and $A=45°$, find a, B and C.

➡ M186

[Sol] $a^2=2^2+(\sqrt{6}+\sqrt{2})^2-2\cdot 2\cdot(\sqrt{6}+\sqrt{2})\cos 45°$

$=8$

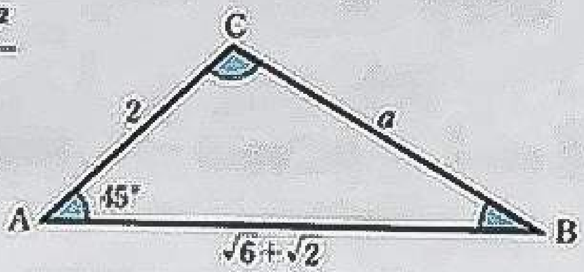
Since $a>0$, $a=2\sqrt{2}$

$$\cos B=\frac{(\sqrt{6}+\sqrt{2})^2+(2\sqrt{2})^2-2^2}{2\cdot(\sqrt{6}+\sqrt{2})\cdot 2\sqrt{2}}$$

$$=\frac{\sqrt{3}}{2}$$

$\therefore B=30°$

$\therefore C=180°-(45°+30°)=105°$



Alternative Solution

$a^2=2^2+(\sqrt{6}+\sqrt{2})^2-2\cdot 2\cdot(\sqrt{6}+\sqrt{2})\cos 45°$

$=8$

Since $a>0$, $a=2\sqrt{2}$

$$\frac{2\sqrt{2}}{\sin 45°}=\frac{2}{\sin B}$$

$$\therefore \sin B=\frac{2\sin 45°}{2\sqrt{2}}$$

$$=\frac{1}{2}$$

Since $A=45°$, $0°<B<135°$; therefore, $B=30°$

$\therefore C=180°-(45°+30°)=105°$

Area of Triangles

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Given that the area of $\triangle ABC$ is S, prove that $S=\frac{1}{2}bc\sin A$.

[Sol] Let h be the height from side AB of $\triangle ABC$ to vertex C.

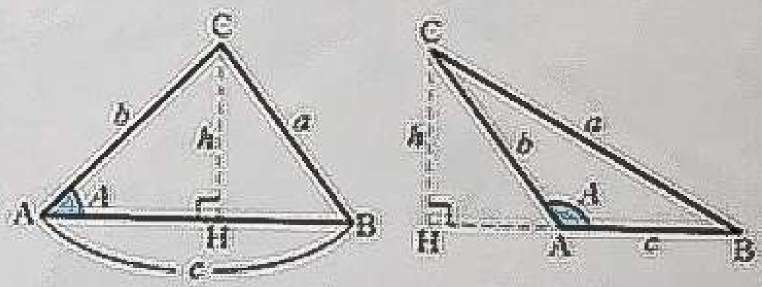
$$h=\boxed{b\sin A}$$

Therefore,

$$S=\frac{1}{2}ch$$

$$=\frac{1}{2}c\boxed{b\sin A}$$

$$=\frac{1}{2}bc\sin A$$



Answers: All the answers are the same. $b\sin A$

Similarly, $S=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$ is true.

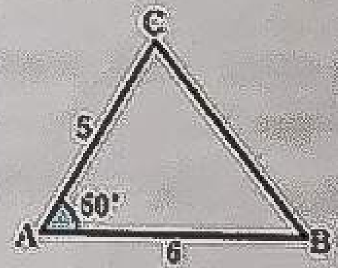
Area of a Triangle

$$S=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$$

1. Given $\triangle ABC$ where $b=5$, $c=6$ and $A=60^\circ$, find its area S.

[Sol] $S=\frac{1}{2}\cdot 5\cdot 6\sin 60^\circ$ ⟵ $S=\frac{1}{2}bc\sin A$

$$=\frac{15\sqrt{3}}{2}$$



2. Given $\triangle ABC$ where $b=3$, $c=4$ and $A=120^\circ$, find its area S.

[Sol] $S=\frac{1}{2}\cdot 3\cdot 4\sin 120^\circ$

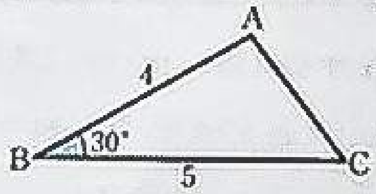
$$=3\sqrt{3}$$

M191b

3. Given $\triangle ABC$ where $a=5$, $c=4$ and $B=30^\circ$, find its area S.

[Sol] $S=\frac{1}{2}\cdot 4\cdot 5\sin 30^\circ$ $\boxed{S=\frac{1}{2}ca\sin B}$

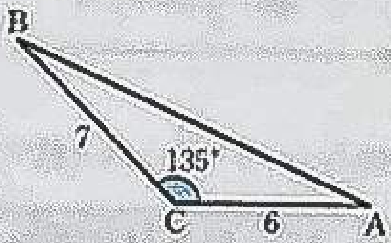
$=5$



4. Given $\triangle ABC$ where $a=7$, $b=6$ and $C=135^\circ$, find its area S.

[Sol] $S=\frac{1}{2}\cdot 7\cdot 6\sin 135^\circ$ $\boxed{S=\frac{1}{2}ab\sin C}$

$=\frac{21\sqrt{2}}{2}$

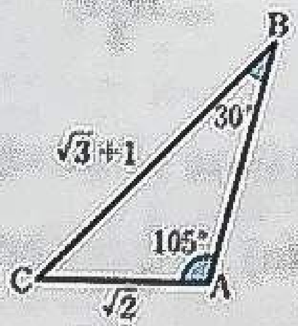


5. Given $\triangle ABC$ where $a=\sqrt{3}+1$, $b=\sqrt{2}$, $A=105^\circ$ and $B=30^\circ$, find its area S.

[Sol] $C=180^\circ-(105^\circ+30^\circ)=45^\circ$

$\therefore S=\frac{1}{2}\cdot(\sqrt{3}+1)\cdot\sqrt{2}\sin 45^\circ$

$=\frac{\sqrt{3}+1}{2}$



Area of Triangles

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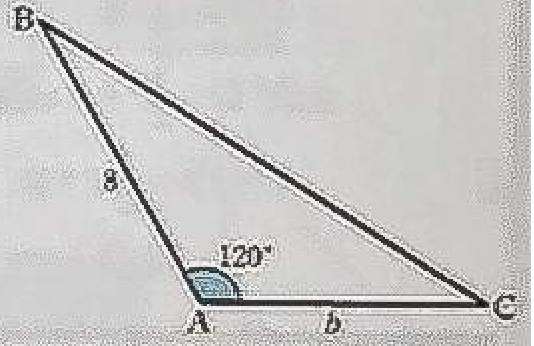
1. Given $\triangle ABC$ where $c=8$, $A=120°$ and area $14\sqrt{3}$, find the value of b.

[Sol] $14\sqrt{3}=\dfrac{1}{2}\cdot b\cdot 8\sin 120°$

$\therefore\ 14\sqrt{3}=2\sqrt{3}b$

$\therefore\ b=7$



2. Given $\triangle ABC$ where $a=2\sqrt{2}$, $B=150°$ and area $\sqrt{3}$, find the value of c.

[Sol] $\sqrt{3}=\dfrac{1}{2}\cdot c\cdot 2\sqrt{2}\sin 150°$

$\therefore\ \sqrt{3}=\dfrac{\sqrt{2}}{2}c$

$\therefore\ c=\sqrt{6}$

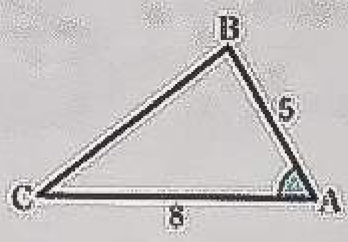
3. Given $\triangle ABC$ where $b=8$, $c=5$ and area $10\sqrt{3}$, find the value of A.

[Sol] $10\sqrt{3}=\dfrac{1}{2}\cdot 8\cdot 5\sin A$

$\therefore\ 10\sqrt{3}=20\sin A$

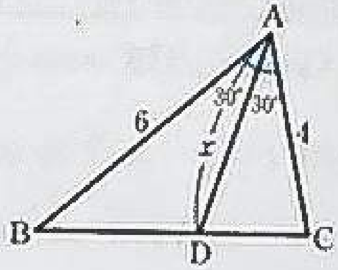
$\sin A=\dfrac{\sqrt{3}}{2}$

$\therefore\ A=60°,\ 120°$



M192b

EX Given $\triangle ABC$ where $AB=6$, $AC=4$, $A=60°$ and point D is the point of intersection of the bisector of angle A and side BC, find the length x of AD.



[Sol] $\triangle ABC=\frac{1}{2}\cdot 6\cdot 4\sin 60°=6\sqrt{3}$

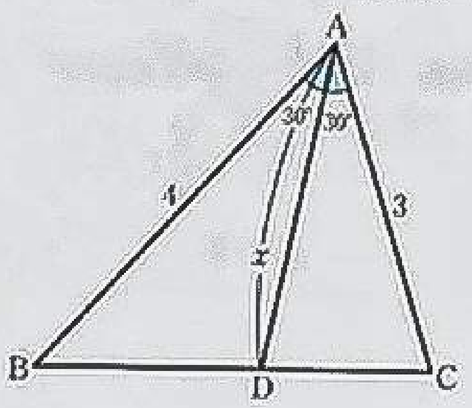
$\triangle ABD=\frac{1}{2}\cdot 6\cdot x\sin 30°=\frac{3}{2}x$

$\triangle ACD=\frac{1}{2}\cdot 4\cdot x\sin 30°=x$

$\therefore\ 6\sqrt{3}=\frac{3}{2}x+x$ ← $\boxed{\triangle ABC=\triangle ABD+\triangle ACD}$

$\therefore\ x=\frac{12\sqrt{3}}{5}$

4. Given $\triangle ABC$ where $AB=4$, $AC=3$, $A=60°$ and point D is the point of intersection of the bisector of angle A and side BC, find the length x of AD.



[Sol] $\triangle ABC=\frac{1}{2}\cdot 4\cdot 3\sin 60°=3\sqrt{3}$

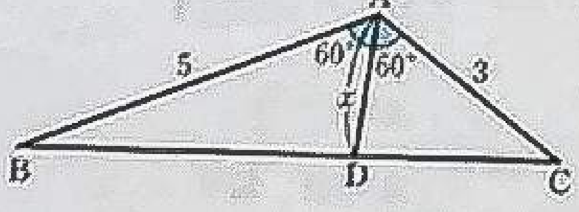
$\triangle ABD=\frac{1}{2}\cdot 4\cdot x\sin 30°=x$

$\triangle ACD=\frac{1}{2}\cdot 3\cdot x\sin 30°=\frac{3}{4}x$

$\therefore\ 3\sqrt{3}=x+\frac{3}{4}x$

$\therefore\ x=\frac{12\sqrt{3}}{7}$

5. Given $\triangle ABC$ where $AB=5$, $AC=3$, $A=120°$ and point D is the point of intersection of the bisector of angle A and side BC, find the length x of AD.



[Sol] $\triangle ABC=\frac{1}{2}\cdot 5\cdot 3\sin 120°=\frac{15\sqrt{3}}{4}$

$\triangle ABD=\frac{1}{2}\cdot 5\cdot x\sin 60°=\frac{5\sqrt{3}}{4}x$

$\triangle ACD=\frac{1}{2}\cdot 3\cdot x\sin 60°=\frac{3\sqrt{3}}{4}x$

$\therefore\ \frac{15\sqrt{3}}{4}=\frac{5\sqrt{3}}{4}x+\frac{3\sqrt{3}}{4}x$

$\therefore\ x=\frac{15}{8}$

Area of Triangles

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Ex. Given $\triangle ABC$ where $a=7$, $b=5$ and $c=4$, find its area S.

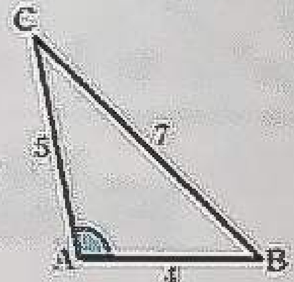
[Sol] $\cos A=\dfrac{5^2+4^2-7^2}{2\cdot5\cdot4}=-\dfrac{1}{5}$ ⟵ $\boxed{\cos A=\dfrac{b^2+c^2-a^2}{2bc}}$

Since $\sin A>0$, ⟵ $\boxed{0^\circ<A<180^\circ}$

$\sin A=\sqrt{1-\left(-\dfrac{1}{5}\right)^2}=\dfrac{2\sqrt{6}}{5}$

$\therefore\ S=\dfrac{1}{2}\cdot5\cdot4\cdot\dfrac{2\sqrt{6}}{5}$ ⟵ $\boxed{S=\dfrac{1}{2}bc\sin A}$

$=4\sqrt{6}$



1. Given $\triangle ABC$ where $a=5$, $b=6$ and $c=7$, find its area S.

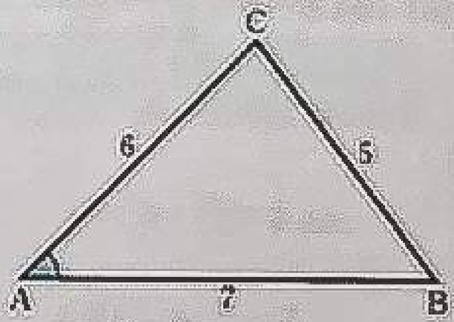
[Sol] $\cos A=\dfrac{6^2+7^2-5^2}{2\cdot6\cdot7}=\dfrac{5}{7}$

Since $\sin A>0$,

$\sin A=\sqrt{1-\left(\dfrac{5}{7}\right)^2}=\dfrac{2\sqrt{6}}{7}$

$\therefore\ S=\dfrac{1}{2}\cdot6\cdot7\cdot\dfrac{2\sqrt{6}}{7}$

$=6\sqrt{6}$



2. Given $\triangle ABC$ where $a=8$, $b=7$ and $c=5$, find its area S.

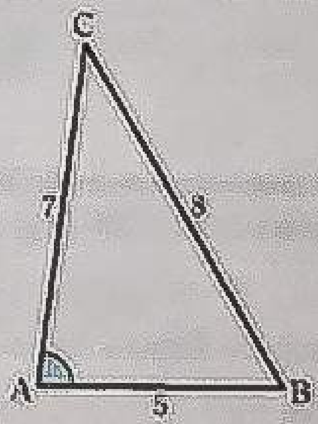
[Sol] $\cos A=\dfrac{7^2+5^2-8^2}{2\cdot7\cdot5}=\dfrac{1}{7}$

Since $\sin A>0$,

$\sin A=\sqrt{1-\left(\dfrac{1}{7}\right)^2}=\dfrac{4\sqrt{3}}{7}$

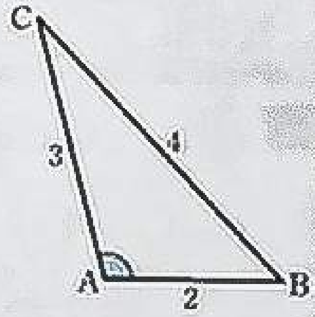
$\therefore\ S=\dfrac{1}{2}\cdot7\cdot5\cdot\dfrac{4\sqrt{3}}{7}$

$=10\sqrt{3}$



M193b

3. Given $\triangle ABC$ where $a=4$, $b=3$ and $c=2$, find its area S.



[Sol] $\cos A=\dfrac{3^2+2^2-4^2}{2\cdot3\cdot2}=-\dfrac{1}{4}$

Since $\sin A>0$,

$$\sin A=\sqrt{1-\left(-\frac{1}{4}\right)^2}=\frac{\sqrt{15}}{4}$$

$$\therefore\ S=\frac{1}{2}\cdot3\cdot2\cdot\frac{\sqrt{15}}{4}$$

$$=\frac{3\sqrt{15}}{4}$$

Alternative Solution

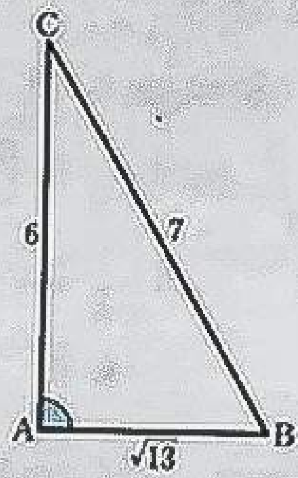
Let $s=\dfrac{a+b+c}{2}$.

$$s=\frac{4+3+2}{2}=\frac{9}{2}$$

$$S=\sqrt{\frac{9}{2}\left(\frac{9}{2}-4\right)\left(\frac{9}{2}-3\right)\left(\frac{9}{2}-2\right)}$$

$$=\frac{3\sqrt{15}}{4}$$

4. Given $\triangle ABC$ where $a=7$, $b=6$ and $c=\sqrt{13}$, find its area S.



[Sol] $\cos A=\dfrac{6^2+(\sqrt{13})^2-7^2}{2\cdot6\cdot\sqrt{13}}=0$

Since $\sin A>0$,

$$\sin A=\sqrt{1-0^2}=1$$

$$\therefore\ S=\frac{1}{2}\cdot6\cdot\sqrt{13}\cdot1$$

$$=3\sqrt{13}$$

The area S of $\triangle ABC$ can also be found by the following formula.

$$S=\sqrt{s(s-a)(s-b)(s-c)},\ \text{where } s=\frac{a+b+c}{2}$$

Using this formula, find the area S of $\triangle ABC$ where $a=7$, $b=5$ and $c=4$.

Since $s=\dfrac{7+5+4}{2}=8$, $S=\sqrt{8(8-7)(8-5)(8-4)}=4\sqrt{6}$

This formula is called ***Heron's Formula***.

Area of Triangles

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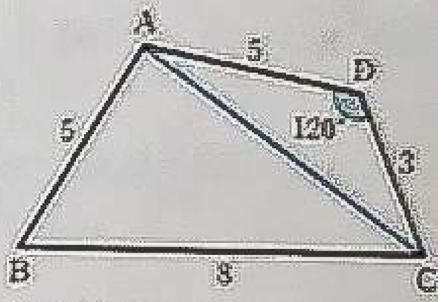
1 Given quadrilateral ABCD where AB=5, BC=8, CD=3, DA=5 and $\angle ADC=120^\circ$, solve the following questions.



(1) Find AC.

[Sol] $AC^2=5^2+3^2-2\cdot5\cdot3\cos120^\circ$

$=49$

Since $AC>0$, $AC=7$

$AC^2=DA^2+CD^2-2\cdot DA\cdot CD\cos\angle ADC$

(2) Find $\angle ABC$.

[Sol] From (1),

$$\cos\angle ABC=\frac{5^2+8^2-7^2}{2\cdot5\cdot8}=\frac{1}{2}$$

$\cos\angle ABC=\dfrac{AB^2+BC^2-AC^2}{2\cdot AB\cdot BC}$

$\therefore\ \angle ABC=60^\circ$

(3) Find the area S of quadrilateral ABCD.

[Sol] From (2),

$$\triangle ABC=\frac{1}{2}\cdot5\cdot8\sin60^\circ=10\sqrt{3}$$

$\triangle ABC=\dfrac{1}{2}\cdot AB\cdot BC\sin\angle ABC$

Also, $\triangle ACD=\dfrac{1}{2}\cdot5\cdot3\sin120^\circ=\dfrac{15\sqrt{3}}{4}$

$\therefore\ S=10\sqrt{3}+\dfrac{15\sqrt{3}}{4}$

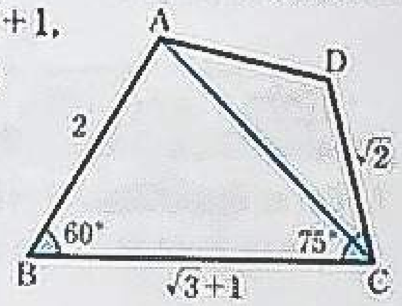
$\triangle ACD=\dfrac{1}{2}\cdot DA\cdot CD\sin\angle ADC$

$=\dfrac{55\sqrt{3}}{4}$

Quadrilateral ABCD $=\triangle ABC+\triangle ACD$

M194b

2. Given quadrilateral ABCD where $AB=2$, $BC=\sqrt{3}+1$, $CD=\sqrt{2}$, $\angle ABC=60°$ and $\angle BCD=75°$, solve the following questions.



(1) Find AC.

[Sol] $AC^2=2^2+(\sqrt{3}+1)^2-2\cdot 2\cdot(\sqrt{3}+1)\cos 60°$

$=6$

Since $AC>0$, $AC=\sqrt{6}$

$AC^2=AB^2+BC^2-2\cdot AB\cdot BC\cos\angle ABC$

(2) Find $\angle ACB$.

[Sol] From (1),

$$\cos\angle ACB=\frac{(\sqrt{6})^2+(\sqrt{3}+1)^2-2^2}{2\cdot\sqrt{6}\cdot(\sqrt{3}+1)}=\frac{\sqrt{2}}{2}$$

$\therefore\ \angle ACB=45°$

$$\cos\angle ACB=\frac{AC^2+BC^2-AB^2}{2\cdot AC\cdot BC}$$

(3) Find the area S of quadrilateral ABCD.

[Sol] $\triangle ABC=\frac{1}{2}\cdot 2\cdot(\sqrt{3}+1)\sin 60°=\frac{3+\sqrt{3}}{2}$

Also, in $\triangle ACD$,

from (2),

$\triangle ABC=\frac{1}{2}\cdot AB\cdot BC\sin\angle ABC$

$\angle ACD=75°-45°=30°$ ⟵ $\angle ACD=\angle BCD-\angle ACB$

$\therefore\ \triangle ACD=\frac{1}{2}\cdot\sqrt{6}\cdot\sqrt{2}\sin 30°=\frac{\sqrt{3}}{2}$ ⟵ $\triangle ACD=\frac{1}{2}\cdot AC\cdot CD\sin\angle ACD$

$\therefore\ S=\frac{3+\sqrt{3}}{2}+\frac{\sqrt{3}}{2}$ ⟵ Quadrilateral $ABCD=\triangle ABC+\triangle ACD$

$=\frac{3+2\sqrt{3}}{2}$

Area of Triangles

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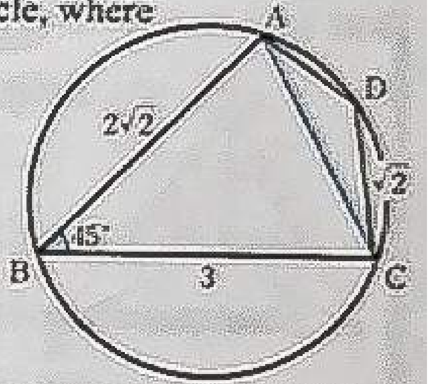
1. Given that quadrilateral ABCD is inscribed in a circle, where $AB=2\sqrt{2}$, $BC=3$, $CD=\sqrt{2}$ and $\angle ABC=45°$, find the area S of quadrilateral ABCD.



[Sol] In $\triangle ABC$,

$$AC^2=\boxed{(2\sqrt{2})^2+3^2-2\cdot 2\sqrt{2}\cdot 3\cos 45°}$$

$$=\boxed{5}$$

$AC^2=AB^2+BC^2-2\cdot AB\cdot BC\cos\angle ABC$

Since $AC>0$, $AC=\boxed{\sqrt{5}}$

Since quadrilateral ABCD is inscribed in a circle,

$$\angle ADC=\boxed{180°}-45°=\boxed{135°}$$

Quadrilateral Inscribed in the Circle (M181)

In $\triangle ACD$, let $AD=x$.

$$(\boxed{\sqrt{5}})^2=\boxed{x^2+(\sqrt{2})^2-2\cdot x\cdot\sqrt{2}\cos 135°}$$

$$\therefore \boxed{x^2+2x-3}=0$$

$$(\boxed{x+3})(\boxed{x-1})=0$$

$$\therefore x=\boxed{-3},\ \boxed{1}$$

Since $x>0$, $x=\boxed{1}$

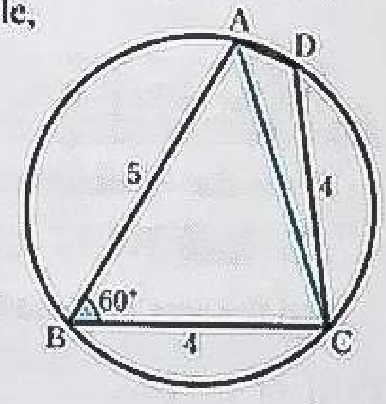
$$\therefore S=\boxed{\frac{1}{2}\cdot 2\sqrt{2}\cdot 3\sin 45°+\frac{1}{2}\cdot\sqrt{2}\cdot 1\cdot\sin 135°}$$

$$=\boxed{\frac{7}{2}}$$

$S=\triangle ABC+\triangle ACD$

M195b

2. Given that quadrilateral ABCD is inscribed in a circle, where AB=5, BC=4, CD=4 and $\angle ABC=60^\circ$, find the area S of quadrilateral ABCD.



[Sol] In $\triangle ABC$,

$$AC^2=5^2+4^2-2\cdot 5\cdot 4\cos 60^\circ$$
$$=21$$

Since $AC>0$, $AC=\sqrt{21}$

Since quadrilateral ABCD is inscribed in a circle,

$$\angle ADC=180^\circ-60^\circ=120^\circ$$

In $\triangle ACD$, let $AD=x$.

$$(\sqrt{21})^2=x^2+4^2-2\cdot x\cdot 4\cos 120^\circ$$

$$\therefore\ x^2+4x-5=0$$
$$(x+5)(x-1)=0$$
$$\therefore\ x=-5,\ 1$$

Since $x>0$, $x=1$

$$\therefore\ S=\frac{1}{2}\cdot 5\cdot 4\sin 60^\circ+\frac{1}{2}\cdot 4\cdot 1\cdot \sin 120^\circ$$
$$=6\sqrt{3}$$

Area of Triangles

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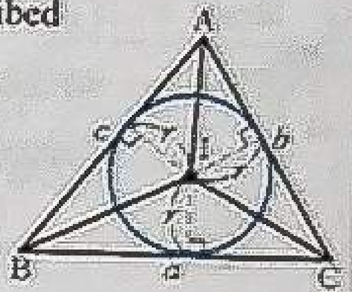
Area of a Triangle with an Inscribed Circle

Let S be the area of $\triangle ABC$, I be the center of the inscribed circle, and r be the radius.

$$S = \triangle IBC + \triangle ICA + \triangle IAB$$
$$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$$
$$= \frac{1}{2}r(a+b+c)$$



Ex. Given $\triangle ABC$ where $a=7$, $b=5$ and $c=6$, find the radius r of the inscribed circle of $\triangle ABC$.

[Sol] $\cos A = \frac{5^2+6^2-7^2}{2\cdot5\cdot6} = \frac{1}{5}$ ← $\boxed{\cos A = \frac{b^2+c^2-a^2}{2bc}}$

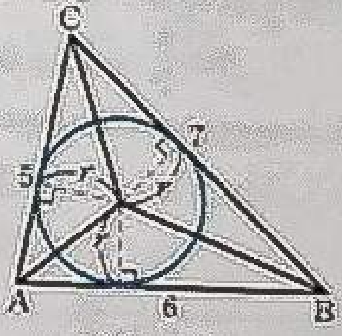
Since $\sin A > 0$, $\sin A = \sqrt{1-\left(\frac{1}{5}\right)^2} = \frac{2\sqrt{6}}{5}$

Therefore, the area S of $\triangle ABC$ is

$$S = \frac{1}{2}\cdot5\cdot6\cdot\frac{2\sqrt{6}}{5} = 6\sqrt{6}$$

Also, $6\sqrt{6} = \frac{1}{2}r(7+5+6)$ ← $\boxed{S = \frac{1}{2}r(a+b+c)}$

$$\therefore\ r = \frac{2\sqrt{6}}{3}$$



1. Given $\triangle ABC$ where $a=3$, $b=7$ and $c=8$, find the radius r of the inscribed circle of $\triangle ABC$.

[Sol] $\cos A = \frac{7^2+8^2-3^2}{2\cdot7\cdot8} = \frac{13}{14}$

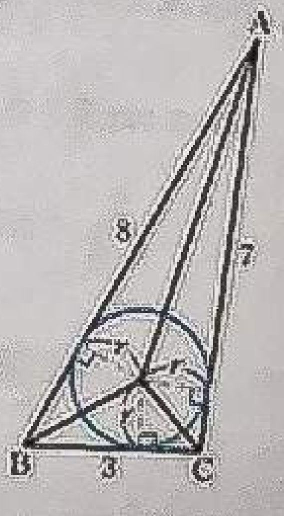
Since $\sin A > 0$, $\sin A = \sqrt{1-\left(\frac{13}{14}\right)^2} = \frac{3\sqrt{3}}{14}$

Therefore, the area S of $\triangle ABC$ is

$$S = \frac{1}{2}\cdot7\cdot8\cdot\frac{3\sqrt{3}}{14} = 6\sqrt{3}$$

Also, $6\sqrt{3} = \frac{1}{2}r(3+7+8)$

$$\therefore\ r = \frac{2\sqrt{3}}{3}$$



M196b

2. Given $\triangle ABC$ where $a=7$, $b=3$ and $c=5$, find the radius r of the inscribed circle of $\triangle ABC$.

[Sol] $\cos A=\dfrac{3^2+5^2-7^2}{2\cdot3\cdot5}=-\dfrac{1}{2}$

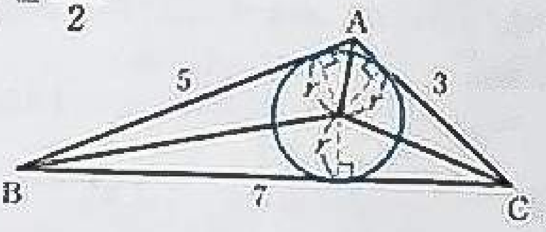
Since $\sin A>0$, $\sin A=\sqrt{1-\left(-\dfrac{1}{2}\right)^2}=\dfrac{\sqrt{3}}{2}$

Therefore, the area S of $\triangle ABC$ is

$$S=\frac{1}{2}\cdot3\cdot5\cdot\frac{\sqrt{3}}{2}=\frac{15\sqrt{3}}{4}$$

Also, $\dfrac{15\sqrt{3}}{4}=\dfrac{1}{2}r(7+3+5)$

$\therefore\ r=\dfrac{\sqrt{3}}{2}$



3. Given $\triangle ABC$ where $b=8$, $c=5$ and $A=60°$, find the radius r of the inscribed circle of $\triangle ABC$.

[Sol] The area S of $\triangle ABC$ is

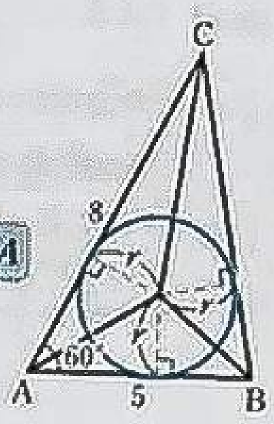
$$S=\frac{1}{2}\cdot8\cdot5\sin60°=10\sqrt{3}$$

Also, $a^2=8^2+5^2-2\cdot8\cdot5\cos60°=49$ ⟵ $\boxed{a^2=b^2+c^2-2bc\cos A}$

Since $a>0$, $a=7$

$\therefore\ 10\sqrt{3}=\dfrac{1}{2}r(7+8+5)$

$\therefore\ r=\sqrt{3}$



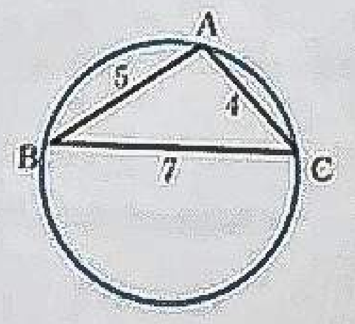
4. Given $\triangle ABC$ where $a=7$, $b=4$ and $c=5$, find the radius R of the circumscribed circle of $\triangle ABC$.

[Sol] $\cos A=\dfrac{4^2+5^2-7^2}{2\cdot4\cdot5}=-\dfrac{1}{5}$

Since $\sin A>0$, $\sin A=\sqrt{1-\left(-\dfrac{1}{5}\right)^2}=\dfrac{2\sqrt{6}}{5}$

$\therefore\ \dfrac{7}{\frac{2\sqrt{6}}{5}}=2R$ ⟵ $\boxed{\dfrac{a}{\sin A}=2R}$

$\therefore\ R=\dfrac{35\sqrt{6}}{24}$



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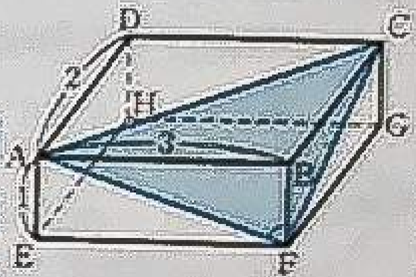
Ex. As shown in the diagram below, given cuboid ABCDEFGH where $AB=3$, $AD=2$ and $AE=1$, find the area S of $\triangle AFC$.



[Sol] $AF=\sqrt{3^2+1^2}=\sqrt{10}$ ← $AF^2=AB^2+BF^2$

$CF=\sqrt{2^2+1^2}=\sqrt{5}$ ← $CF^2=BC^2+BF^2$

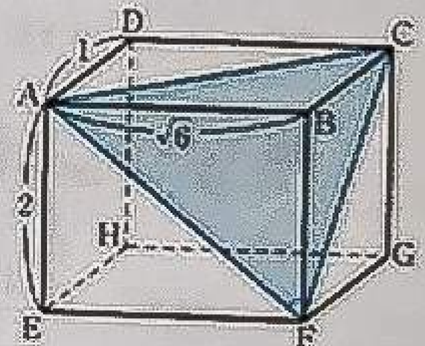
$AC=\sqrt{3^2+2^2}=\sqrt{13}$ ← $AC^2=AB^2+BC^2$

$$\cos\angle AFC=\frac{(\sqrt{10})^2+(\sqrt{5})^2-(\sqrt{13})^2}{2\cdot\sqrt{10}\cdot\sqrt{5}}=\frac{\sqrt{2}}{10} \leftarrow \cos\angle AFC=\frac{AF^2+CF^2-AC^2}{2\cdot AF\cdot CF}$$

Since $\sin\angle AFC>0$, $\sin\angle AFC=\sqrt{1-\left(\frac{\sqrt{2}}{10}\right)^2}=\frac{7\sqrt{2}}{10}$

$$\therefore S=\frac{1}{2}\cdot\sqrt{10}\cdot\sqrt{5}\cdot\frac{7\sqrt{2}}{10}=\frac{7}{2}$$

1. As shown in the diagram below, given cuboid ABCDEFGH where $AB=\sqrt{6}$, $AD=1$ and $AE=2$, find the area S of $\triangle AFC$.



[Sol] $AF=\sqrt{(\sqrt{6})^2+2^2}=\sqrt{10}$

$CF=\sqrt{1^2+2^2}=\sqrt{5}$

$AC=\sqrt{(\sqrt{6})^2+1^2}=\sqrt{7}$

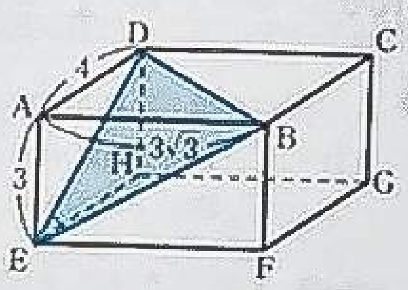
$$\cos\angle AFC=\frac{(\sqrt{10})^2+(\sqrt{5})^2-(\sqrt{7})^2}{2\cdot\sqrt{10}\cdot\sqrt{5}}=\frac{2\sqrt{2}}{5}$$

Since $\sin\angle AFC>0$, $\sin\angle AFC=\sqrt{1-\left(\frac{2\sqrt{2}}{5}\right)^2}=\frac{\sqrt{17}}{5}$

$$\therefore S=\frac{1}{2}\cdot\sqrt{10}\cdot\sqrt{5}\cdot\frac{\sqrt{17}}{5}=\frac{\sqrt{34}}{2}$$

M197b

2. As shown in the diagram below, given cuboid ABCDEFGH where $AB=3\sqrt{3}$, $AD=4$ and $AE=3$, find the area S of $\triangle BDE$.



[Sol] $BD=\sqrt{(3\sqrt{3})^2+4^2}=\sqrt{43}$

$DE=\sqrt{4^2+3^2}=5$

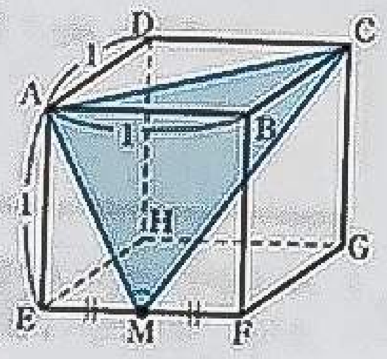
$BE=\sqrt{(3\sqrt{3})^2+3^2}=6$

$\cos\angle BED=\dfrac{5^2+6^2-(\sqrt{43})^2}{2\cdot 5\cdot 6}=\dfrac{3}{10}$

Since $\sin\angle BED>0$, $\sin\angle BED=\sqrt{1-\left(\dfrac{3}{10}\right)^2}=\dfrac{\sqrt{91}}{10}$

$\therefore\ S=\dfrac{1}{2}\cdot 5\cdot 6\cdot\dfrac{\sqrt{91}}{10}=\dfrac{3\sqrt{91}}{2}$

3. As shown in the diagram, given cube ABCDEFGH where the length of each side is 1 and M is the midpoint of side EF, find the area S of $\triangle AMC$.



[Sol] $AM=\sqrt{1^2+\left(\dfrac{1}{2}\right)^2}=\dfrac{\sqrt{5}}{2}$

$AC=\sqrt{1^2+1^2}=\sqrt{2}$

$MC=\sqrt{1^2+1^2+\left(\dfrac{1}{2}\right)^2}=\dfrac{3}{2}$ ⟵ Since $\angle CFM=90^\circ$ and $CF^2=BF^2+BC^2$, $MC=\sqrt{CF^2+MF^2}=\sqrt{BF^2+BC^2+MF^2}$

$\cos\angle AMC=\dfrac{\left(\dfrac{\sqrt{5}}{2}\right)^2+\left(\dfrac{3}{2}\right)^2-(\sqrt{2})^2}{2\cdot\dfrac{\sqrt{5}}{2}\cdot\dfrac{3}{2}}=\dfrac{\sqrt{5}}{5}$

Since $\sin\angle AMC>0$, $\sin\angle AMC=\sqrt{1-\left(\dfrac{\sqrt{5}}{5}\right)^2}=\dfrac{2\sqrt{5}}{5}$

$\therefore\ S=\dfrac{1}{2}\cdot\dfrac{\sqrt{5}}{2}\cdot\dfrac{3}{2}\cdot\dfrac{2\sqrt{5}}{5}=\dfrac{3}{4}$

A cuboid with six equal square faces is called a *cube*.

Area of Triangles

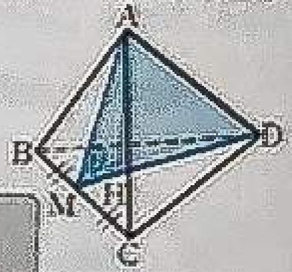
100%	~90%	~80%	~70%	69%~

Name

Date / /

Time : to :

1. There is a regular tetrahedron ABCD where each side has length 6 (AB=AC=AD=BC=BD=CD=6). Given that M is the midpoint of side BC, $\angle AMD=\theta$ and AH is the perpendicular dropped from vertex A to MD, solve the following questions.

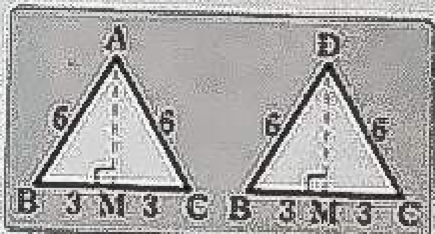


(1) Find the value of $\cos\theta$.

[Sol] Since $\triangle ABM$ and $\triangle BDM$ are right-angled triangles,

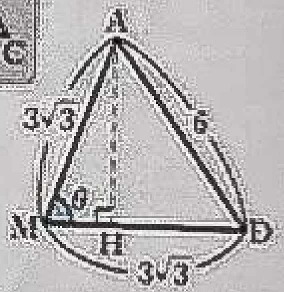
$$AM=\sqrt{\boxed{6^2-3^2}}=\boxed{3\sqrt{3}}$$

$$DM=\sqrt{\boxed{6^2-3^2}}=\boxed{3\sqrt{3}}$$



Therefore, in $\triangle AMD$,

$$\cos\theta=\frac{(\boxed{3\sqrt{3}})^2+(\boxed{3\sqrt{3}})^2-\boxed{6}^2}{2\cdot\boxed{3\sqrt{3}}\cdot\boxed{3\sqrt{3}}}=\boxed{\frac{1}{3}}$$



(2) Find the length h of AH.

[Sol] Since $\sin\theta>0$, from (1),

$$\sin\theta=\sqrt{1-\left(\boxed{\frac{1}{3}}\right)^2}=\boxed{\frac{2\sqrt{2}}{3}}$$

$$\therefore h=\boxed{3\sqrt{3}\cdot\frac{2\sqrt{2}}{3}}=\boxed{2\sqrt{6}} \longleftarrow \boxed{h=AM\sin\theta}$$

(3) Find the volume V of regular tetrahedron ABCD.

[Sol] Let S be the area of $\triangle BCD$.

$$S=\boxed{\frac{1}{2}\cdot6\cdot3\sqrt{3}}=\boxed{9\sqrt{3}} \longleftarrow \boxed{S=\frac{1}{2}\cdot BC\cdot DM}$$

$$\therefore V=\boxed{\frac{1}{3}\cdot9\sqrt{3}\cdot2\sqrt{6}}=\boxed{18\sqrt{2}} \longleftarrow \boxed{V=\frac{1}{3}Sh}$$

A triangular pyramid with four equilateral triangle faces is called a ***regular tetrahedron***.
Let V be the volume of a pyramid or a cone with base area S and height h. Then, $V=\frac{1}{3}Sh$.

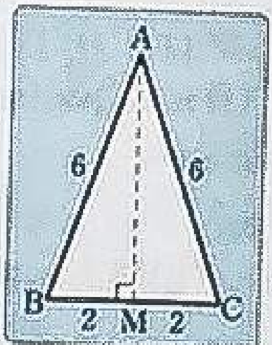
M198b

2. There is a tetrahedron ABCD where $AB=AC=AD=6$ and $BC=CD=DB=4$. Given that M is the midpoint of side BC, $\angle AMD=\theta$ and AH is the perpendicular dropped from vertex A to MD, solve the following questions.

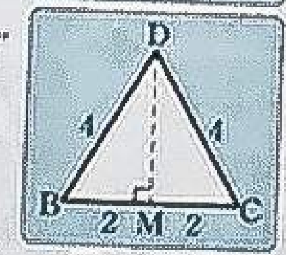
(1) Find the value of $\cos\theta$.

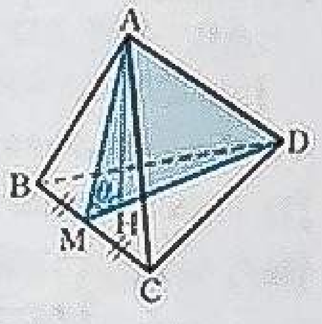
[Sol] Since $\triangle ABM$ and $\triangle BDM$ are right-angled triangles,

$$AM=\sqrt{6^2-2^2}=4\sqrt{2}$$



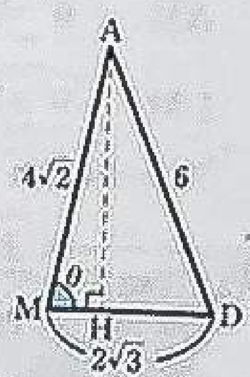
$$DM=\sqrt{4^2-2^2}=2\sqrt{3}$$





Therefore, in $\triangle AMD$,

$$\cos\theta=\frac{(4\sqrt{2})^2+(2\sqrt{3})^2-6^2}{2\cdot 4\sqrt{2}\cdot 2\sqrt{3}}=\frac{\sqrt{6}}{12}$$



(2) Find the length h of AH.

[Sol] Since $\sin\theta>0$, from (1),

$$\sin\theta=\sqrt{1-\left(\frac{\sqrt{6}}{12}\right)^2}=\frac{\sqrt{138}}{12}$$

$$\therefore\ h=4\sqrt{2}\cdot\frac{\sqrt{138}}{12}=\frac{2\sqrt{69}}{3}$$

(3) Find the volume V of tetrahedron ABCD.

[Sol] Let S be the area of $\triangle BCD$.

$$S=\frac{1}{2}\cdot 4\cdot 2\sqrt{3}=4\sqrt{3}$$

$$\therefore\ V=\frac{1}{3}\cdot 4\sqrt{3}\cdot\frac{2\sqrt{69}}{3}=\frac{8\sqrt{23}}{3}$$

KUMON

Area of Triangles

Name

100%	~90%	~80%	~70%	69%~

Date / /

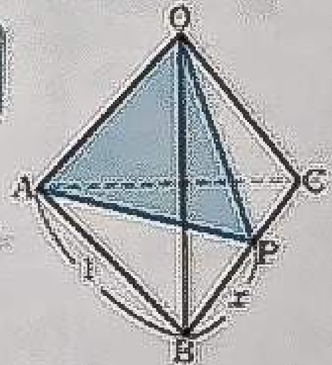
Time : to :

1. Given a regular tetrahedron OABC where each side has length 1, place point P on side BC and let the length of line segment BP be x.

(1) Express the area of triangle OAP in terms of x.

[Sol] In $\triangle ABP$, since $AB=1$ and $\angle ABP=60°$,

$$AP^2=1^2+x^2-2\cdot1\cdot x\cos60° \quad \leftarrow \boxed{\text{Cosine Rule (M184)}}$$
$$=x^2-x+1 \cdots ①$$

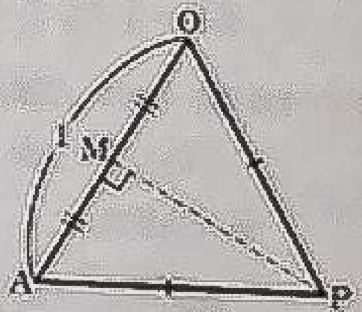


In $\triangle OBP$, since $OB=1$ and $\angle OBP=60°$,

$$OP^2=1^2+x^2-2\cdot1\cdot x\cos60°$$
$$=x^2-x+1 \cdots ②$$

From ① and ②, $AP=OP$

Therefore, $\triangle OAP$ is an isosceles triangle with $AP=OP$. Dropping a perpendicular PM from P to side OA,



$$PM=\sqrt{AP^2-AM^2}=\sqrt{x^2-x+\frac{3}{4}}$$

Thus,

$$\triangle OAP=\frac{1}{2}\cdot1\cdot\sqrt{x^2-x+\frac{3}{4}} \quad \leftarrow \boxed{\triangle OAP=\frac{1}{2}\cdot OA\cdot PM}$$
$$=\frac{1}{2}\sqrt{x^2-x+\frac{3}{4}}$$

(2) Find the minimum value of the area of triangle OAP for which point P moves along side BC.

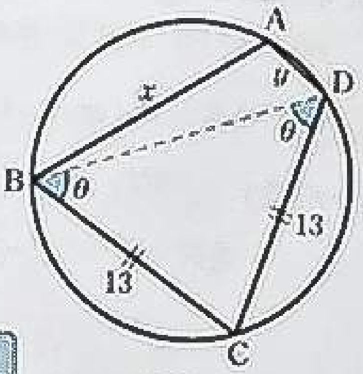
[Sol] From (1), $\triangle OAP=\frac{1}{2}\sqrt{\left(x-\frac{1}{2}\right)^2+\frac{1}{2}}$

Since $0\leqq x\leqq 1$,

the minimum value is $\frac{\sqrt{2}}{4}$, at $x=\frac{1}{2}$.

M199b

2. Quadrilateral ABCD is inscribed in a circle with radius $\frac{65}{8}$. Given that the perimeter of quadrilateral ABCD is 44 and the lengths of sides BC and CD are 13 each, find the lengths of the remaining two sides AB and DA.



[Sol] Let AB$=x$ and DA$=y$.

$$x+y=44-13\cdot 2=18 \quad \cdots ①$$

In $\triangle$BCD, since BC$=$CD,

let $\angle$CBD$=\angle$CDB$=\theta$.

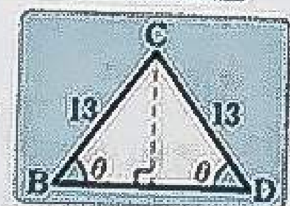
$$\frac{13}{\sin\theta}=2\cdot\frac{65}{8} \quad \Leftarrow \quad \frac{a}{\sin A}=2R$$

$$\therefore \sin\theta=\frac{4}{5}$$

Since $0^\circ<\theta<90^\circ$, $\Leftarrow$ In $\triangle$BCD, $2\theta+\angle\mathrm{BCD}=180^\circ$, $\theta=\frac{180^\circ-\angle\mathrm{BCD}}{2}<90^\circ$

$$\cos\theta=\sqrt{1-\left(\frac{4}{5}\right)^2}=\frac{3}{5}$$

$$\mathrm{BD}=2\cdot 13\cdot\frac{3}{5}=\frac{78}{5} \quad \cdots ② \quad \Leftarrow$$



Also, since $\angle\mathrm{BCD}=180^\circ-2\theta$,

$$\angle\mathrm{BAD}=180^\circ-(180^\circ-2\theta)=2\theta \quad \Leftarrow \quad \boxed{\text{Quadrilateral Inscribed in the Circle (M181)}}$$

In $\triangle$ABD,

$$\mathrm{BD}^2=x^2+y^2-2xy\cos 2\theta \quad \Leftarrow \quad \boxed{\text{Cosine Rule (M184)}}$$

$$=x^2+y^2-2xy(1-2\sin^2\theta) \quad \Leftarrow \quad \boxed{\cos 2\theta=1-2\sin^2\theta}$$

$$=x^2+y^2+\frac{14}{25}xy \quad \Leftarrow \quad \boxed{\sin\theta=\frac{4}{5}}$$

$$=(x+y)^2-\frac{36}{25}xy$$

$$=324-\frac{36}{25}xy \quad \cdots ③ \quad \Leftarrow \quad \boxed{\text{From ①, } x+y=18}$$

From ② and ③, $xy=56$ $\cdots$④ $\Leftarrow$ $\left(\frac{78}{5}\right)^2=324-\frac{36}{25}xy$

From ① and ④, x and y are two solutions of $t^2-18t+56=0$. $\Leftarrow$

Therefore, since $(t-4)(t-14)=0$, $t=4,\ 14$

Thus, $x=4,\ y=14$ or $x=14,\ y=4$

$\therefore$ AB$=4$, DA$=14$ or AB$=14$, DA$=4$

Root-Coefficient Relationships (J147)

Area of Triangles

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~

1. Given $\triangle ABC$ where $AB=3$, $AC=4$, $A=120°$ and point D is the point of intersection of the bisector of angle A and side BC, find the length x of AD

➡ M192

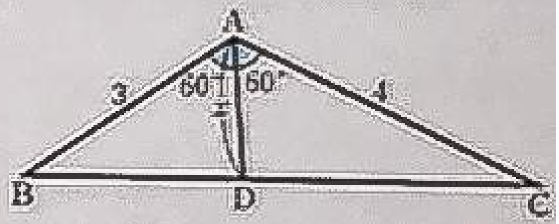
[Sol] $\triangle ABC=\frac{1}{2}\cdot 3\cdot 4\sin 120°=3\sqrt{3}$

$$\triangle ABD=\frac{1}{2}\cdot 3\cdot x\sin 60°=\frac{3\sqrt{3}}{4}x$$

$$\triangle ACD=\frac{1}{2}\cdot 4\cdot x\sin 60°=\sqrt{3}x$$

$$\therefore 3\sqrt{3}=\frac{3\sqrt{3}}{4}x+\sqrt{3}x$$

$$\therefore x=\frac{12}{7}$$



2. Given $\triangle ABC$ where $a=6$, $b=4$ and $c=8$, find its area S.

➡ M193

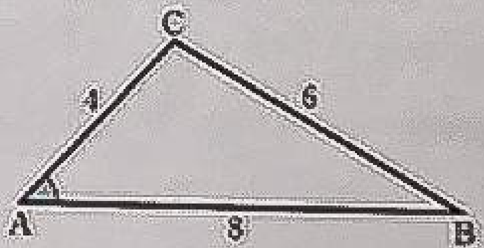
[Sol] $\cos A=\frac{4^2+8^2-6^2}{2\cdot 4\cdot 8}=\frac{11}{16}$

Since $\sin A>0$,

$$\sin A=\sqrt{1-\left(\frac{11}{16}\right)^2}=\frac{3\sqrt{15}}{16}$$

$$\therefore S=\frac{1}{2}\cdot 4\cdot 8\cdot\frac{3\sqrt{15}}{16}$$

$$=3\sqrt{15}$$



Alternative Solution

Let $s=\frac{a+b+c}{2}$.

$$s=\frac{6+4+8}{2}=9$$

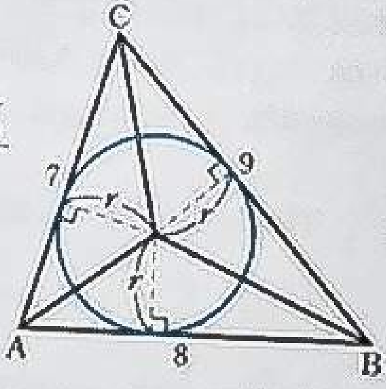
$$S=\sqrt{9(9-6)(9-4)(9-8)}$$

$$=3\sqrt{15}$$

3. Given $\triangle ABC$ where $a=9$, $b=7$ and $c=8$, find the radius r of the inscribed circle of $\triangle ABC$.

➡ M196

[Sol] $\cos A=\dfrac{7^2+8^2-9^2}{2\cdot7\cdot8}=\dfrac{2}{7}$



Since $\sin A>0$, $\sin A=\sqrt{1-\left(\dfrac{2}{7}\right)^2}=\dfrac{3\sqrt{5}}{7}$

Therefore, the area S of $\triangle ABC$ is

$$S=\frac{1}{2}\cdot7\cdot8\cdot\frac{3\sqrt{5}}{7}=12\sqrt{5}$$

Also, $12\sqrt{5}=\dfrac{1}{2}r(9+7+8)$

$\therefore\ r=\sqrt{5}$

Alternative Solution

Let $s=\dfrac{a+b+c}{2}$.

$s=\dfrac{9+7+8}{2}=12$

$S=\sqrt{12(12-9)(12-7)(12-8)}$
$=12\sqrt{5}$

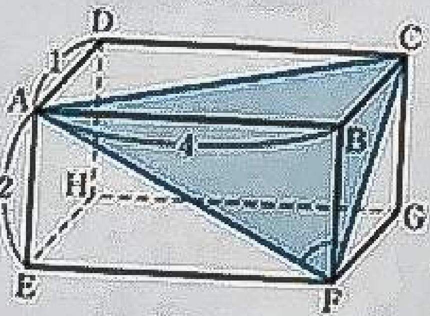
Also, $12\sqrt{5}=\dfrac{1}{2}r(9+7+8)$

$\therefore\ r=\sqrt{5}$

4. As shown in the diagram below, given cuboid ABCDEFGH where AB=4, AD=1 and AE=2, find the area S of $\triangle AFC$.

➡ M197

[Sol] $AF=\sqrt{4^2+2^2}=2\sqrt{5}$



$CF=\sqrt{1^2+2^2}=\sqrt{5}$

$AC=\sqrt{4^2+1^2}=\sqrt{17}$

$$\cos\angle AFC=\frac{(2\sqrt{5})^2+(\sqrt{5})^2-(\sqrt{17})^2}{2\cdot2\sqrt{5}\cdot\sqrt{5}}=\frac{2}{5}$$

Since $\sin\angle AFC>0$, $\sin\angle AFC=\sqrt{1-\left(\dfrac{2}{5}\right)^2}=\dfrac{\sqrt{21}}{5}$

$$\therefore\ S=\frac{1}{2}\cdot2\sqrt{5}\cdot\sqrt{5}\cdot\frac{\sqrt{21}}{5}=\sqrt{21}$$

Mathematics Solution Book M

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